CS 252

F, 10 May 2024

$$T(n) \leq T(n-1) + n \; ; \quad T(1) = 1$$

Guess

$$T(2) = T(1) + 2 = 3 \qquad 1+2$$

$$T(3) = T(2) + 3 = 6 \qquad 1+2+3$$

$$T(4) = T(3) + 4 = 10 \qquad 1+2+3+4$$

$$\vdots$$

$$T(n) \leq \frac{n(n+1)}{2}$$

Prove by induction $T(n) \leq \frac{n(n+1)}{2}$

Base case: $T(1) = 1 \leq \frac{1(1+1)}{2} = 1$ ✓ P_1 is true

Engine

Proposition parameterized by k

P_k : $T(k) \leq \frac{k(k+1)}{2}$

If P_k is true, then P_{k+1} must be true

If P_k, then

$$T(k) \le \frac{k(k+1)}{2}$$

so

$$T(k+1) \le T(k) + k + 1$$

$$\le \frac{k(k+1)}{2} + k + 1$$

$$= \frac{(k+1)(k+2)}{2}$$

ie. P_{k+1} is true

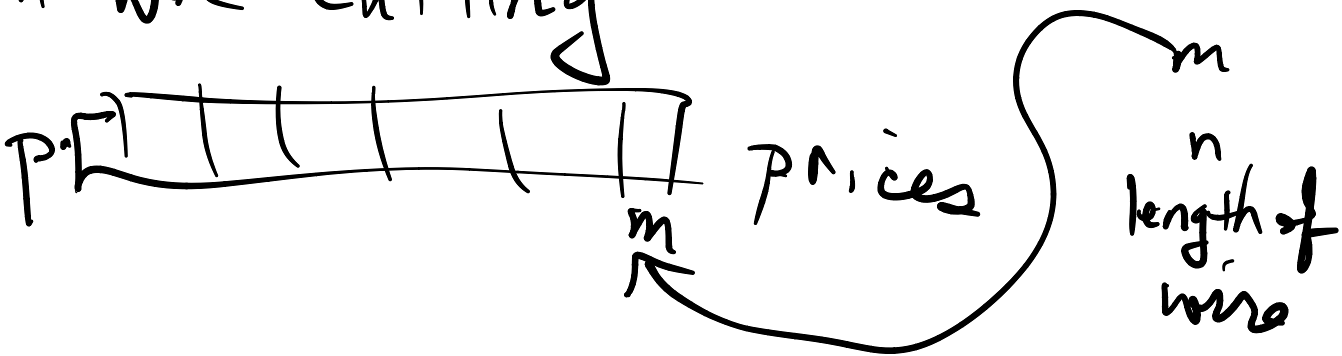
I've shown

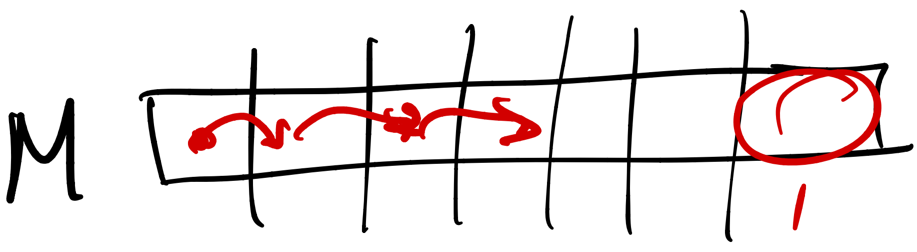
P_1 is true

$P_k \Rightarrow P_{k+1} \quad \forall k$

done.

Recall wire-cutting





M[n] was the answer

Knapsack problem

	0	1	2	3	4	5	6	7
sizes	2	5	1	1	3	1	8	6
values	7	6	1	2	4	9	10	6

capacity 12

Goal: maximize total value
keep total size ≤ 12

Robbing the house

Backpack can hold total size of 12.

What do you steal?

Take the diamonds, not the piano

	0	1	2	3	4	5	6	7
sizes	2	5	1	1	3	1	8	6
values	7	6	1	2	4	9	10	6

capacity 12

Let $M(k, C)$ = the max value you can get with objects $0 \ldots k$ & capacity C

Goal $M(7, 12)$

	0	1	2	3	4	5	6	7
sizes	2	5	1	1	3	1	8	6
values	7	6	1	2	4	9	10	6

capacity 12

$$M(7,12) = \max \begin{cases} 6 + M(6,6) \\ M(6,12) \end{cases}$$

$$M(k, C) = \max \begin{cases} \text{values}[k] + M(k-1, C - \text{size}[k]) \\ M(k-1, C) \end{cases}$$

Either we include item k in our haul, or we don't.

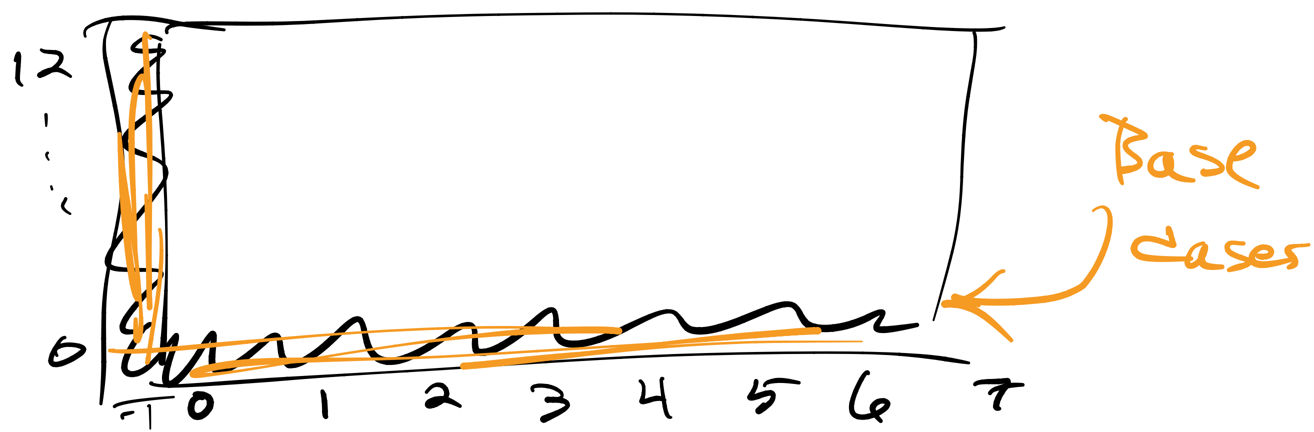
Base case?

	0	1	2	3	4	5	6	7
sizes	2	5	1	1	3	1	8	6
values	7	6	1	2	4	9	10	6

capacity 12

$$M(k, C) = \max \begin{cases} \text{values}[k] + M(k-1, C - \text{size}[k]) \\ M(k-1, C) \end{cases}$$

Base case



	0	1	2	3	4	5	6	7
sizes	2	5	1	1	3	1	8	6
values	7	6	1	2	4	9	10	6

Capacity 12

IF $C \geq size[k]$

$$M(k, C) = \max \begin{cases} values[k] + M(k-1, C - size[k]) \\ M(k-1, C) \end{cases}$$

$M(k,c)$

k \ c	0	1	2	3	4	5	6	7	8	9	10	11	12
-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	7	7	7	7	7	7	7	7	7	7	7
1	0												
2	0												
3	0												
4	0												
5	0												
6	0												
7	0												

$$M(0,1) = \max\big(M(-1,1),\ \cancel{\text{value}[0] + M(-1,-2)}\big)$$

$$\cancel{7 + M(-1,-2)}$$

uh-oh

$$M(k,c) = \max\big(M(k-1,c),\ \underbrace{\text{value}[k] + M(k-1, c-\text{size}[k])}_{\text{if } c \geq \text{size}[k]}\big)$$

$M(k, c)$

c

K	0	1	2	3	4	5	6	7	8	9	10	11	12
-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	7	7	7	7	7	7	7	7	7	7	7
1	0	0	7	7	7	7	7	13	13	13	13	13	13
2	0												
3	0												
4	0												
5	0												
6	0												
7	0												

How long?

items × max capacity

$$M(k,c) = \max\left(M(k-1,c),\ \underbrace{value[k] + M(k-1, c-size[k])}_{\text{if } c \geq size[k]}\right)$$