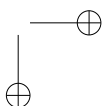
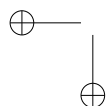
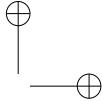
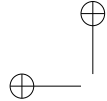


# Connecting Discrete Mathematics and Computer Science

David Liben-Nowell



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REVISED EDITION

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**David Liben-Nowell**

Department of Computer Science

Carleton College



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**To MDSWM, with never-ending appreciation, and in loving memory of my grandfather, Jay Liben, who brought more joy, curiosity, and kvetching to this world than anyone else I know.**

# Contents

<i>List of Computer Science Connections</i>	<i>page x</i>
<i>Acknowledgements</i>	xiii
<i>Credits</i>	xvi
<b>1 On the Point of this Book</b>	1-1
<b>2 Basic Data Types</b>	2-1
2.1 Why You Might Care	2-2
2.2 Booleans, Numbers, and Arithmetic	2-4
Exercises	2-24
2.3 Sets: Unordered Collections	2-28
Exercises	2-44
2.4 Sequences, Vectors, and Matrices: Ordered Collections	2-48
Exercises	2-65
2.5 Functions	2-70
Exercises	2-87
2.6 Chapter at a Glance	2-90
<b>3 Logic</b>	3-1
3.1 Why You Might Care	3-2
3.2 An Introduction to Propositional Logic	3-4
Exercises	3-19
3.3 Propositional Logic: Some Extensions	3-22
Exercises	3-36
3.4 An Introduction to Predicate Logic	3-40
Exercises	3-60
3.5 Predicate Logic: Nested Quantifiers	3-63
Exercises	3-75
3.6 Chapter at a Glance	3-80

<b>4 Proofs</b>	4-1
4.1 Why You Might Care	4-2
4.2 Error-Correcting Codes	4-4
Exercises	4-25
4.3 Proofs and Proof Techniques	4-30
Exercises	4-50
4.4 Some Examples of Proofs	4-53
Exercises	4-72
4.5 Common Errors in Proofs	4-75
Exercises	4-84
4.6 Chapter at a Glance	4-89
<b>5 Mathematical Induction</b>	5-1
5.1 Why You Might Care	5-2
5.2 Proofs by Mathematical Induction	5-4
Exercises	5-24
5.3 Strong Induction	5-28
Exercises	5-40
5.4 Recursively Defined Structures and Structural Induction	5-45
Exercises	5-58
5.5 Chapter at a Glance	5-61
<b>6 Analysis of Algorithms</b>	6-1
6.1 Why You Might Care	6-2
6.2 Asymptotics	6-4
Exercises	6-18
6.3 Asymptotic Analysis of Algorithms	6-22
Exercises	6-37
6.4 Recurrence Relations: Analyzing Recursive Algorithms	6-40
Exercises	6-57
6.5 An Extension: Recurrence Relations of the Form $T(n) = aT\left(\frac{n}{b}\right) + cn^k$	6-60
Exercises	6-70
6.6 Chapter at a Glance	6-72

## viii Contents

<b>7</b>	<b>Number Theory</b>	7-1
7.1	Why You Might Care	7-2
7.2	Modular Arithmetic	7-4
	Exercises	7-18
7.3	Primality and Relative Primality	7-21
	Exercises	7-40
7.4	Multiplicative Inverses	7-43
	Exercises	7-55
7.5	Cryptography	7-58
	Exercises	7-69
7.6	Chapter at a Glance	7-72
<b>8</b>	<b>Relations</b>	8-1
8.1	Why You Might Care	8-2
8.2	Formal Introduction	8-4
	Exercises	8-21
8.3	Properties of Relations: Reflexivity, Symmetry, and Transitivity	8-24
	Exercises	8-42
8.4	Special Relations: Equivalence Relations and Partial/Total Orders	8-45
	Exercises	8-63
8.5	Chapter at a Glance	8-66
<b>9</b>	<b>Counting</b>	9-1
9.1	Why You Might Care	9-2
9.2	Counting Unions and Sequences	9-4
	Exercises	9-28
9.3	Using Functions to Count	9-33
	Exercises	9-50
9.4	Combinations and Permutations	9-56
	Exercises	9-77
9.5	Chapter at a Glance	9-82
<b>10</b>	<b>Probability</b>	10-1
10.1	Why You Might Care	10-2



	<b>Contents</b>	<b>ix</b>
10.2 Probability, Outcomes, and Events		10-6
Exercises		10-26
10.3 Independence and Conditional Probability		10-30
Exercises		10-52
10.4 Random Variables and Expectation		10-57
Exercises		10-82
10.5 Chapter at a Glance		10-87
<b>11 Graphs and Trees</b>		<b>11-1</b>
11.1 Why You Might Care		11-2
11.2 Formal Introduction		11-4
Exercises		11-30
11.3 Paths, Connectivity, and Distances		11-34
Exercises		11-53
11.4 Trees		11-57
Exercises		11-75
11.5 Weighted Graphs		11-79
Exercises		11-92
11.6 Chapter at a Glance		11-95
<b>12 Looking Forward</b>		<b>12-1</b>
Index		12-5
References		12-15

# List of Computer Science Connections

## *Chapter 1: On the Point of this Book*

## *Chapter 2: Basic Data Types*

Integers and ints, Reals and floats	2-20
Computing Square Roots, and Not Computing Square Roots	2-22
Set Building in Languages (and MapReduce)	2-40
Clustering Data Sets (and Speech Processing)	2-42
The Vector Space Model	2-61
Rotation Matrices	2-63
Hash Tables and Hash Functions	2-85

## *Chapter 3: Logic*

Natural Language Processing, Ambiguity, and Truth	3-17
Computational Complexity, Satisfiability, and a Million Dollars	3-32
Short-Circuit Evaluation, Optimization, and Modern Compilers	3-34
Game Trees, Logic, and Winning Tic-Tac(-Toe)	3-54
Nonlocal Variables and Lexical vs. Dynamic Scoping	3-56
Gödel's Incompleteness Theorem	3-58
Currying	3-73

## *Chapter 4: Proofs*

Reed–Solomon Codes	4-23
Paul Erdős, “The Book,” and Erdős Numbers	4-46
The Four-Color Theorem (and Massive Computer-Generated Proofs)	4-48
Cryptography and the Generation of Prime Numbers	4-68
Other Uncomputable Problems (That You Might Care About)	4-70
The Cost of Missing Proofs: Some Famous Bugs in CS	4-82

## *Chapter 5: Mathematical Induction*

## List of Computer Science Connections xi

Loop Invariants	5-20
Onion Routing, Steganography, and Conversations whose Very Existence is Sensitive	5-22
Triangulation, Computer Graphics, and 3D Surfaces	5-36
Max Heaps	5-38
Grammars, Parsing, and Ambiguity	5-56
 <i>Chapter 6: Analysis of Algorithms</i>	
Vacuum Tubes, Transistors, and Moore’s Law	6-16
Multitasking, Garbage Collection, and Wall Clocks	6-33
Time, Space, and Complexity	6-35
AVL Trees	6-53
Divide-and-Conquer Algorithms and Matrix Multiplication	6-68
 <i>Chapter 7: Number Theory</i>	
Converting Between Bases, Binary Representation, and Generating Strings	7-16
Secret Sharing	7-36
Error Correction with Reed–Solomon Codes	7-38
Miller–Rabin Primality Testing	7-53
Diffie–Hellman Key Exchange	7-67
 <i>Chapter 8: Relations</i>	
Relational Databases	8-17
Automating Decisions, Facial Recognition, and Algorithmic Bias	8-19
What’s Hard about Designing Voting Systems	8-38
Regular Expressions	8-40
Deterministic Finite Automata (DFAs)	8-59
The Painter’s Algorithm and Hidden-Surface Removal	8-61
 <i>Chapter 9: Counting</i>	
Running out of IP addresses, and IPv6	9-22
A Lower Bound for Comparison-Based Sorting	9-24
Infinite Cardinalities (and Problems that Can’t Be Solved by Any Program)	9-46
Lossy and Lossless Compression	9-48
Brute Force Algorithms and Dynamic Programming	9-73
The Enigma Machine and a First Computer	9-75

**xii List of Computer Science Connections***Chapter 10: Probability*

Using Randomness to Protect Privacy	10-20
Quantum Computing	10-22
Information, Charles Dickens, and the Entropy of English	10-24
Speech Recognition, Bayes' Rule, and Language Models	10-48
Bayesian Modeling and Spam Filtering	10-50
The Monte Carlo Method	10-76
A Randomized Algorithm for Finding Medians	10-78

*Chapter 11: Graphs and Trees*

Degree Distributions and the Heavy Tail	11-26
Graph Drawing, Graph Layouts, and the 9/11 Memorial	11-28
The Bowtie Structure of the Web	11-49
Garbage Collection	11-51
Directed Graphs, Cycles, and Kidney Transplants	11-71
Binary Search Trees	11-73
Random Walks and Ranking Web Pages	11-90

*Chapter 12: Looking Forward*

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Would thou hadst less deserved,  
That the proportion both of thanks and payment  
Might have been mine! only I have left to say,  
More is thy due than more than all can pay.

---

William Shakespeare (1564–1616)  
The Scottish Play (c. 1606)

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David Liben-Nowell  
Northfield, MN  
August 2021

PS: I would be delighted to receive any comments or suggestions. Please don't hesitate to get in touch.

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