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DEPARTMENT OF COMPUTER SCIENCE

CARLETON COLLEGE

DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

OR

(A BIT OF) THE MATH THAT COMPUTER SCIENTISTS NEED TO KNOW

*To MDSWM, with never-ending appreciation, and
in loving memory of my grandfather, Jay Liben, who
brought more joy, curiosity, and kvetching to this
world than anyone else I know.*

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Acknowledgements

Would thou hadst less deserved,
That the proportion both of thanks and payment
Might have been mine! only I have left to say,
More is thy due than more than all can pay.

William Shakespeare (1564–1616)
The Scottish Play

To everyone who has helped, directly and indirectly, with everything over these last years—these words cannot adequately convey my thanks, but at least they’re a start: *thank you!*

I owe special thanks to a very long list of generous and warm people—many more than I can mention here—for advice and kindness and support, both technical and emotional, as this book came into being. For those whom I haven’t named by name, please know that it’s only because I have gotten such great support from so many people, and I hope that you’ll consider this sentence the promise that, when we next see each other, the first round’s on me. While I’m leaving out the names of the many people who have helped make my life happy and fulfilling while I’ve been working on this book, I do want to give specific thanks to a few people:

I want to thank my colleagues—near and far, including many who are not just colleagues but also dear friends and beloved family members—for their wisdom and patience, for answering my endlessly annoying questions, and for conversations that led to examples or exercises or bug fixes or the very existence of this entire book (even if you didn’t know that’s what we were talking about at the time): Eric Alexander, Tanya Berger-Wolf, Kelly Connole, Amy Csizmar Dalal, Josh Davis, Roger Downs, Laura Effinger-Dean, Eric Egge, Adriana Estill, Andy Exley, Alex Freeman, Sherri Goings, Jack Goldfeather, Deanna Haunsperger, Pierre Hecker, David Huyck, Sue Jandro, Sarah Jansen, Iris Jastram, Jon Kleinberg, Carissa Knipe, Mark Krusemeyer, Jessica Leiman, Lynn Liben, Jadrian Miles, Dave Musicant, Gail Nelson, Rich Nowell, Layla Oesper, Jeff Ondich, Sam Patterson, Anna Rafferty, Alexa Sharp, Julia Strand, Mike Tie, Zach Weinersmith, Tom Wexler, Kevin Woods, Jed Yang, and Steve Zdancewic.

I also owe my appreciation to Don Fowley, Bryan Gambrel, Beth Golub, Jessy Moor, Anna Pham, Sondra Scott, and Gladys Soto at Wiley. Thanks to Judy Brody for relentless and efficient pursuit of permissions (from many different people and publishers)

to use the quotes that appear as epigraphs throughout the book. And thanks as well to the many insightful reviewers of previous drafts of this material. So many times I got chapter reviews back and put them aside in a huff, only to come back to the reviewers' comments months later and realize that their suggestions were exactly right. (And, to be clear: blame me, not them, for the errors that I'm sure remain.)

I specifically want to thank Eric Alexander, Laura Biester, Josh Davis, Charlotte Foran, Jadrian Miles, Dave Musicant, Layla Oesper, Anna Rafferty, Jed Yang, and the Carleton CS 202 students from 2013–2017 for their willingness to work with early, and buggy, drafts of this book. And thanks to those and many other students at Carleton for their patience, and for sending their comments and suggestions for improvements—in particular: Hami Abdi, David Abel, Alexander Auyeung, Andrew Bacon, Kharmen Bharucha, John Blake, Caleb Braun, Macallan Brown, Adam Canady, Noah Carnahan, Yitong Chen, Jinny Cho, Leah Cole, Katja Collier, Lila Conlee, Eric Ewing, Greg Fournier, Andy Freeland, Emma Freeman, Samuel Greaves, Reilly Hallstrom, Jacob Hamalian, Sylvie Hauser, Jack Hessel, Joy Hill, Matt Javalay, Emily Johnston, Emily Kampa, Carlton Keedy, Henry Keiter, Jonathan Knudson, Julia Kroll, Brennan Kuo, Edward Kwiatkowski, Dimitri Lang, Tristan Leigh, Zach Levonian, Daniel Levy, Rhys Lindmark, Gordon Loery, David Long, Robert Lord, Inara Makhmudova, Elliot Mawby, Javier Moran Lemus, Sean Mullan, Micah Nacht, Justin Norden, Laurel Orr, Raven Pillmann, Josh Pitkofsky, Matthew Pruyne, Nikki Rhodes, Will Schifeling, Colby Seyferth, Alex Simonides, Oscar Smith, Kyung Song, Frederik Stensaeth, Patrick Stephen, Maximiliano Villarreal, Alex Voorhees, Allie Warren, Ben Wedin, Michael Wheatman, Jack Wines, Christopher Winter, and Andrew Yang.

This book would not have been possible without the support of Carleton College, not only for the direct support of this project, but also for providing a wonderfully engaging place to make my professional home. When I started at Carleton, my friends and family back east thought that moving to Minnesota (the frontier!) was nothing less than a sign that I had finally lost it, and I have to admit that I thought they had a point. But it's been a fabulous place to have landed, with great friends and colleagues and students—the kind who don't let you get away with anything, but in a good way.

Some of the late stages of the work on this book occurred while I was visiting the University of Cambridge. Thanks to Churchill College and the Computer Laboratory, and especially to Melissa Hines and Cecilia Mascolo, for their hospitality and support.

And my thanks to the somewhat less formal host institutions that have fueled this writing: Brick Oven Bakery, Cakewalk, Goodbye Blue Monday, Tandem Bagels, The Hideaway (Northfield, MN); Anodyne, Blue Moon, Bull Run, Caffetto, Common Roots, Espresso Royale, Isles Bun & Coffee, Keen Eye, Plan B, Precision Grind, Reverie, Spyhouse, Sebastian Joe's, The Beat, The Nicollet, The Purple Onion, Turtle Bread Company, Uncommon Grounds, Urban Bean (Minneapolis, MN); Ginkgo, Grand Central, Kopplin's (St. Paul, MN); Collegetown Bagels (Ithaca, NY); Slave to the Grind (Bronxville, NY); Bloc Eleven, Diesel Cafe (Somerville, MA); Lyndell's (Cambridge, MA); Tryst (Washington, DC); Hot Numbers, Espresso Library (Cambridge, England); and various Starbucks, Caribous, and Dunn Brothers.

And, last but certainly not least, my deepest gratitude to my friends and family for all your help and support while this project has consumed both hours and years. You know who you are, and I hope you also know how much I appreciate you. *Thank you!*

David Liben-Nowell
Northfield, MN
May 2017

PS: I would be delighted to receive any comments or suggestions from readers. Please don't hesitate to get in touch.

Credits

This book was typeset using L^AT_EX, and I produced all but a few figures from scratch using a combination of PSTricks and TikZ. The other figures are reprinted with permission from their copyright holders. The illustrations that open every chapter were drawn by Carissa Knipe (<http://carissaknipe.com>), who was a complete delight to work with—both on these illustrations and when she was a student at Carleton. I took the photograph of a house in Figure 2.48 myself. Figure 4.5 (the Therac-25 diagram) is reproduced from Nancy Leveson’s book *Safeware: System Safety and Computers* with permission from Pearson Education. Figure 4.27 (a poem proving the undecidability of the Halting Problem) is reproduced with permission from Geoffrey K. Pullum. Figure 5.22 (triangulations of a rabbit) is reproduced from a paper by Tobias Isenberg, Knut Hartmann, and Henry König with permission from the Society for Modeling and Simulation International (SCS). Figure 11.15 (a map of some European train routes) is reproduced with permission from RGBAlpha/Getty Images.¹

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