Math 5707 Stable Matchings

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"It is a truth universally acknowledged, that a single man in possession of a good fortune must be in want of a wife."

Recall 1. A binary relation \( \leq \) is a linear or total order if it is antisymmetric \((a \leq b \text{ and } b \leq a \text{ implies } a = b)\), transitive, and total \((a \leq b \text{ or } b \leq a)\).

Definition 2. Let \( G = (A, B, E) \) be a bipartite graph. For each \( v \in A \sqcup B \), let \( \leq_v \) be a linear order on \( N(v) \). Call the collection \( \{\leq_v\}_{v \in V} \) a set of preferences for \( G \). A matching \( M \) of \( G \) is stable if for every edge \( e \in E \setminus M \), there exists an edge \( f \in M \) such that \( e = vx, f = vy \), and \( x <_v y \).

Example 3. National Resident Matching Program

Definition 4 (Stable matching algorithm). Let \( G = (A, B, E) \) be a bipartite graph and \( \{\leq_v\}_{v \in V} \) a set of preferences.

(a) For each vertex \( a \in A \), let \( b \in N(a) \) be the \( \leq_a \)-maximal vertex, and add \( ab \) to \( M \).

(b) For each vertex \( b \in B \) incident to multiple edges in \( M \), let \( a \in N(b) \) be the \( \leq_b \)-maximal vertex such that \( ab \in M \), and delete from \( E \) (and thus also from \( M \)) all edges in \( M \) incident to \( b \) except for \( ab \).

(c) Repeat the steps above until unmatched \( a \in A \) are all isolated.

"And now nothing remains for me but to assure you in the most animated language of the violence of my affection."

Theorem 5 (2.1.4, Gale–Shapley 1962). Given a bipartite graph \( G = (A, B, E) \) and a set of preferences, the stable matching algorithm produces a stable matching \( M \). Moreover, if \( G = K_{n,n} \) is a complete bipartite graph, then \( M \) is perfect.

Proof. Note that if \( b \in B \) is in \( M \) after some round, it will always be in \( M \). Furthermore, \( b \) will only “trade up” and be matched with increasingly more desirable vertices. Therefore we will never get the same \( M \) again (except immediately, when we terminate). As there are only finitely many possible \( M \), the process will terminate at some point.

Suppose \( ab \in E(G) \setminus M \). If \( a \) never proposed to \( b \), that means \( a \) is currently matched with someone \( b' \in B \) more preferable \((b <_a b')\). Otherwise, \( a \) proposed to \( b \) at some point. Then \( ab \) was deleted from \( M \) at some point, which could only happen if \( b \) had a better match \( a' \in A \) available. Since \( b \) only trades up, \( b \) is currently matched with someone \( a'' \in A \) more preferable. □

Exercise 6. Stable matchings might not exist in non-bipartite graphs. For example, a triangle where each vertex prefers its right neighbour.

Example 7. Stable matchings are not (necessarily) unique. For example, a 4-cycle where each vertex prefers its neighbour on the right. This also means whether \( A \) or \( B \) propose will lead to different results in the algorithm.