Math 4990 Problem Set 9

Due Tuesday, Nov 10, 2015 in class

Please refer to previous problem sets for instructions, including but not limited to the collaboration policy.

ASSIGNMENT

Let $T_n$ be the number of domino tilings of a $2 \times n$ region. Recall the linear recurrence relation

$$T_n = T_{n-1} + T_{n-2}$$

for $n \geq 3$. Using $[\text{7}]$ for $n = 2$ suggests that it is sensible to define $T_0 = 1$. For the rest of the problem set, it is advisable to NOT use $[\text{7}]$. That is, each time you see $T_n$, interpret it as “number of tilings” as opposed to the actual number.

Problem 1. Prove that $T_{a+b} = T_a T_b + T_{a-1} T_{b-1}$.

Problem 2. Prove that $\binom{n}{1} T_0 + \binom{n}{2} T_1 + \cdots + \binom{n}{n} T_{n-1} = T_{2n-1}$.
[Hint: Associate a number $i$ to each domino tiling of a $2 \times (2n - 1)$ region, $1 \leq i \leq n$, such that there are $\binom{n}{i} T_{i-1}$ tilings associated with $i$.]

Problem 3. Let $f(n)$ denote the number of domino tilings of a $3 \times n$ region, and let $g(n)$ denote the number of domino tilings of the same region but with one corner square removed.

1. Calculate $f(n)$ and $g(n)$ from their definitions for $1 \leq n \leq 4$.
2. Write $f(n)$ in terms of $f(a)$ and $g(b)$ for some $a, b < n$.
   [Hint: Follow the idea that allowed us to find $[\text{7}]$.]
3. Similarly, write a recurrence relation for $g(n)$ in terms of $f$ and $g$.
4. Using the recurrence relations, define $f(0)$ and $g(0)$ sensibly.
5. Obtain a linear recurrence relation of $f$ alone by eliminating $g$.
6. Let $h(n) = f(2n)$ and calculate $h(n)$ for $0 \leq n \leq 9$.
   [Hint: One way to do this is by using Wolfram Alpha. For example, to get the first ten Fibonacci numbers, go to $\text{http://www.wolframalpha.com}$, enter \begin{verbatim}
f(n)=f(n-1)+f(n-2), f(0)=0, f(1)=1
\end{verbatim}
and click “more” on the result page.]
7. What is the number of domino tilings of a $3 \times 40$ region?
   [Hint: One way to do this is by using the On-Line Encyclopedia of Integer Sequences. Go to $\text{http://oeis.org}$ and search with the first few terms of the sequence.]