Math 4707 Random Graphs

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The aim is to use random graphs to show that graphs with no short cycles can have high chromatic number. These lecture notes roughly follow Diestel's text.

Theorem 1 (Markov's inequality)**.** *Let $X \geq 0$ be a random variable on Ω and $a > 0$. Then*

$$\Pr[X \geq a] \leq \mathbb{E}(X)/a.$$

Proof.

$$\begin{aligned} \mathbb{E}(X) &= \sum_{t \in \Omega} \Pr(t) \cdot X(t) \\ &\geq \sum_{\substack{t \in \Omega \\ X(t) \geq a}} \Pr(t) \cdot X(t) \\ &\geq \sum_{\substack{t \in \Omega \\ X(t) \geq a}} \Pr(t) \cdot a \\ &\geq \Pr[X \geq a] \cdot a. \end{aligned}$$

□

Definition 2. Fix $V = [n]$ and consider all (simple) graphs on vertex set V. For each of the $\binom{n}{2}$ pairs of vertices, decide whether there is an edge in the following manner. Pick $0 \leq p \leq 1$ and let $q = 1 - p$. Independently for each pair of vertices, let the edge be present with probability p and absent with probability q. [There is a way to formally construct a probability space that satisfies this requirement.] Call this space $\mathcal{G}(n, p)$.

Example 3. Let $G \in \mathcal{G}(n, p)$ be sampled. Fix a graph H on $[n]$ with m edges. Then $\Pr[H \subseteq G] = p^m$ and $\Pr[H = G] = p^m q^{\binom{n}{2} - m}$.

Lemma 4. *Let $X : \mathcal{G}(n, p) \to \mathbb{N}$ be the random variable that counts the number of k-cycles. The expected number of k-cycles in $G \in \mathcal{G}(n, p)$ is*

$$\mathbb{E}[X] = \frac{n! p^k}{2k(n-k)!}.$$

Proof. Linearity of expectation; necklaces have $2k$-fold symmetry. □

Definition 5. Let $G = (V, E)$ be a graph. A set $S \subseteq V$ of vertices is **independent** if the vertices in S are pairwise non-adjacent. In other words, the induced graph $G[S]$ is empty and has no edges. The **independence number** of a graph G, denoted $\alpha(G)$, is the maximum size of an independent set.

Lemma 6. *For any graph G, we have $|V(G)| \leq \alpha(G)\chi(G)$.*

Proof. Each colour class is an independent set. □

Lemma 7. *Let $k > 0$ be an integer, and $p = p(n)$ be a function of n such that $p \geq (6k \ln n)/n$ for n large. Sample $G_n \in \mathcal{G}(n, p)$ for each n. Then*

$$\lim_{n\to\infty} \Pr\left[\alpha(G_n) \geq \tfrac{1}{2}n/k\right] = 0.$$

Proof. Let $r = \left\lceil \frac{1}{2}n/k \right\rceil$. By union bound, the probability that G_n has a set of r independent vertices is at most

$$\begin{aligned}\Pr[\alpha(G_n) \geq r] &\leq \tbinom{n}{r} q^{\binom{r}{2}} \\ &\leq n^r q^{\binom{r}{2}} \\ &= (nq^{(r-1)/2})^r \\ &\leq (ne^{-p(r-1)/2})^r,\end{aligned}$$

since $q = 1 - p \leq e^{-p}$. Now for large n, we get

$$\begin{aligned}ne^{-p(r-1)/2} &= ne^{-pr/2+p/2} \\ &\leq ne^{-\frac{3}{2}\ln n + p/2} \\ &\leq n \cdot n^{-3/2} e^{1/2} \\ &= \tfrac{\sqrt{e}}{\sqrt{n}},\end{aligned}$$

which tends to 0 as $n \to \infty$, as desired. □

Definition 8. The **girth** of a graph G, denoted $g(G)$, is the length of a shortest cycle. [By convention, write $g(G) = \infty$ if G is acyclic, and say $\infty > k$ for any integer k.]

Theorem 9 (Erdős 1959)**.** *For any integer k, there exists a graph H with girth $g(H) > k$ and chromatic number $\chi(H) > k$.*

Proof. Assume that $k \geq 3$, fix ε with $0 < \varepsilon < 1/k$, and let $p = n^{\varepsilon - 1}$. Say a cycle is **short** if its length is at most k. Let $X(G)$ be a random variable denoting the number of short cycles in a random graph $G \in \mathcal{G}(n, p)$. By Lemma 4, we get

$$\mathbb{E}[X] = \sum_{i=3}^{k} \frac{n! p^i}{2i(n-i)!} \leq \tfrac{1}{2} \sum_{i=3}^{k} n^i p^i \leq \tfrac{1}{2}(k-2) n^k p^k,$$

where $(np)^i \leq (np)^k$ because $np = n^\varepsilon \geq 1$.

By Markov's inequality (Theorem 1), we get

$$\begin{aligned}\Pr[X \geq n/2] &\leq \mathbb{E}[X]/(n/2) \\ &\leq (k-2) n^{k-1} p^k \\ &= (k-2) n^{k-1} n^{(\varepsilon-1)k} \\ &= (k-2) n^{k\varepsilon - 1}.\end{aligned}$$

Note that $k\varepsilon - 1 < 0$, so

$$\lim_{n\to\infty} \Pr[X \geq n/2] = 0.$$

Note that for large n, $p = n^{\varepsilon-1} \geq (6k \ln n)/n$, so Lemma 7 gives

$$\lim_{n\to\infty} \Pr\left[\alpha \geq \tfrac{1}{2}n/k\right] = 0.$$

Pick n large enough such that $\Pr[X \geq n/2] < \frac{1}{2}$ and $\Pr[\alpha \geq \frac{1}{2}n/k] < \frac{1}{2}$. Then there is some $G \in \mathcal{G}(n, p)$ with fewer than $n/2$ short cycles and $\alpha(G) < \frac{1}{2}n/k$.

Delete a vertex from each of the short cycles to obtain a subgraph H. Since H has no short cycles, $g(H) > k$. Since we deleted fewer than $n/2$ vertices, we get $|V(H)| > n/2$. Also note that $\alpha(H) \leq \alpha(G)$. By Lemma 6, we get

$$\chi(H) \geq \frac{|V(H)|}{\alpha(H)} > \frac{n/2}{\alpha(G)} > k,$$

as desired. □