Math 4707 Random Graphs

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The aim is to use random graphs to show that graphs with no short cycles can have high chromatic number. These lecture notes roughly follow Diestel’s text.

**Theorem 1** (Markov’s inequality). Let $X \geq 0$ be a random variable on $\Omega$ and $a > 0$. Then

$$
\Pr[X \geq a] \leq \frac{E(X)}{a}.
$$

**Proof.**

$$
\begin{align*}
\mathbb{E}(X) &= \sum_{t \in \Omega} \Pr(t) \cdot X(t) \\
&\geq \sum_{t \in \Omega \text{ s.t. } X(t) \geq a} \Pr(t) \cdot X(t) \\
&\geq \sum_{t \in \Omega \text{ s.t. } X(t) \geq a} \Pr(t) \cdot a \\
&\geq \Pr[X \geq a] \cdot a.
\end{align*}
$$

**Definition 2.** Fix $V = [n]$ and consider all (simple) graphs on vertex set $V$. For each of the $\binom{n}{2}$ pairs of vertices, decide whether there is an edge in the following manner. Pick $0 \leq p \leq 1$ and let $q = 1 - p$. Independently for each pair of vertices, let the edge be present with probability $p$ and absent with probability $q$. [There is a way to formally construct a probability space that satisfies this requirement.] Call this space $G(n, p)$.

**Example 3.** Let $G \in G(n, p)$ be sampled. Fix a graph $H$ on $[n]$ with $m$ edges. Then $\Pr[H \subseteq G] = p^m$ and $\Pr[H = G] = p^m q^{\binom{n}{2} - m}$.

**Lemma 4.** Let $X : G(n, p) \rightarrow \mathbb{N}$ be the random variable that counts the number of $k$-cycles. The expected number of $k$-cycles in $G \in G(n, p)$ is

$$
\mathbb{E}[X] = \frac{n! p^k}{2k(n - k)!}.
$$

**Proof.** Linearity of expectation; necklaces have $2k$-fold symmetry.

**Definition 5.** Let $G = (V, E)$ be a graph. A set $S \subseteq V$ of vertices is **independent** if the vertices in $S$ are pairwise non-adjacent. In other words, the induced graph $G[S]$ is empty and has no edges. The **independence number** of a graph $G$, denoted $\alpha(G)$, is the maximum size of an independent set.

**Lemma 6.** For any graph $G$, we have $|V(G)| \leq \alpha(G) \chi(G)$.

**Proof.** Each colour class is an independent set.
Lemma 7. Let $k > 0$ be an integer, and $p = p(n)$ be a function of $n$ such that $p \geq (6k \ln n)/n$ for $n$ large. Sample $G_n \in \mathcal{G}(n, p)$ for each $n$. Then

$$\lim_{n \to \infty} Pr \left[ \alpha(G_n) \geq \frac{1}{2} n/k \right] = 0.$$ 

Proof. Let $r = \left[ \frac{1}{2} n/k \right]$. By union bound, the probability that $G_n$ has a set of $r$ independent vertices is at most

$$Pr[\alpha(G_n) \geq r] \leq \binom{n}{r} q^{(r/2)}$$

$$\leq n^r q^{(r/2)}$$

$$= (np^{(r-1)/2})^r$$

$$\leq (n p^{(r-1)/2})^r,$$

since $q = 1 - p \leq e^{-p}$. Now for large $n$, we get

$$n p^{(r-1)/2} = n p^{r/2 + p/2}$$

$$\leq n p^{-3/2} n + p/2$$

$$\leq n \cdot n^{-3/2} e^{1/2}$$

$$= \sqrt{\frac{e}{\pi}},$$

which tends to 0 as $n \to \infty$, as desired. 

Definition 8. The girth of a graph $G$, denoted $g(G)$, is the length of a shortest cycle. [By convention, write $g(G) = \infty$ if $G$ is acyclic, and say $\infty > k$ for any integer $k$.]

Theorem 9 (Erdős 1959). For any integer $k$, there exists a graph $H$ with girth $g(H) > k$ and chromatic number $\chi(H) > k$.

Proof. Assume that $k \geq 3$, fix $\varepsilon$ with $0 < \varepsilon < 1/k$, and let $p = n^{\varepsilon - 1}$. Say a cycle is short if its length is at most $k$. Let $X(G)$ be a random variable denoting the number of short cycles in a random graph $G \in \mathcal{G}(n, p)$. By Lemma 4 we get

$$E[X] = \sum_{i=3}^{k} \frac{np^i}{2i(n-i)!} \leq \frac{1}{2} \sum_{i=3}^{k} np^i \leq \frac{1}{2} (k - 2) n^k p^k,$$

where $(np)^i \leq (np)^k$ because $np = n^\varepsilon \geq 1$.

By Markov’s inequality (Theorem 1), we get

$$Pr[X \geq n/2] \leq E[X]/(n/2)$$

$$\leq (k - 2) n^{k-1} p^k$$

$$= (k - 2) n^{k-1} n^{(\varepsilon-1)k}$$

$$= (k - 2) n^{k\varepsilon-1}.$$ 

Note that $k\varepsilon - 1 < 0$, so

$$\lim_{n \to \infty} Pr[X \geq n/2] = 0.$$ 

Note that for large $n$, $p = n^{\varepsilon-1} \geq (6k \ln n)/n$, so Lemma 7 gives

$$\lim_{n \to \infty} Pr[\alpha \geq \frac{1}{2} n/k] = 0.$$ 

Pick $n$ large enough such that $Pr[X \geq n/2] < 1/2$ and $Pr[\alpha \geq \frac{1}{2} n/k] < 1/2$. Then there is some $G \in \mathcal{G}(n, p)$ with fewer than $n/2$ short cycles and $\alpha(G) < \frac{1}{2} n/k$.

Delete a vertex from each of the short cycles to obtain a subgraph $H$. Since $H$ has no short cycles, $g(H) > k$. Since we deleted fewer than $n/2$ vertices, we get $|V(H)| > n/2$. Also note that $\alpha(H) \leq \alpha(G)$. By Lemma 6 we get

$$\chi(H) \geq \frac{|V(H)|}{\alpha(H)} \geq \frac{n/2}{\alpha(G)} > k,$$

as desired. 

□