Math 4707 Midterm 1 Practice Questions

Questions courtesy of Jang Soo Kim.

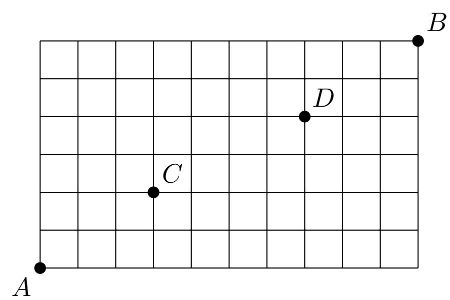
Problem 1. Given a finite set A, subsets $B \subseteq A$, $C \subseteq A$, and suppose B and C are disjoint ($B \cap C = \varnothing$). Find the number of subsets $S \subseteq A$ such that $|S \cap B| = 1$ and $|S \cap C| = 2$.

Problem 2. For an integer t, we define $s(t)$ to be the sum of digits of the binary form of t. [For example, $s(13) = 1+1+0+1 = 3$ as $13 = 1101_2$ in binary.] Find the sum $s(0)+s(1)+s(2)+\ldots+s(511)$ (in decimal).

Problem 3. Find the number of ways to put n indistinguishable balls into k bins labelled $1, 2, \ldots, k$ such that the bin labelled i gets at least i balls.

Problem 4. Find the number of shortest (monotonic) paths on the grid from $A = (0, 0)$ to $B = (10, 6)$ with given conditions.

(a) No condition.
(b) Visit $C = (3, 2)$.
(c) Visit C but not $D = (7, 4)$.
(d) Visit neither C nor D.



Problem 5. What is the number of 5-digit numbers in which at least two digits are equal?

Problem 6. The sequence a_n is given by $a_0 = 0$, $a_1 = 2$, and $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$. Find a formula for a_n.

Problem 7. Prove the following identity for $n \geq 2$:

$$0\binom{n}{0} + 2\binom{n}{2} + 4\binom{n}{4} + \ldots = n \cdot 2^{n-2}.$$

Problem 8. Suppose $n, m, k \in \mathbb{N}$ are positive integers. Prove the following identity:

$$\sum_{i=0}^{n} i\binom{n}{i}\binom{m}{k-i} = n\binom{n+m-1}{k-1}.$$