Math 4707 Midterm 1 Solutions

You may use books, notes, and calculators on this exam. Calculators will not be necessary. Please refrain from using other electronic devices such as laptops or cell phones. Do your work individually without collaboration. You have the full class period for the exam, and may leave early after turning in your work.

Problems 1 through 4 are worth 7 points each. Please write your solutions carefully, showing all your work and justifying your steps rigorously. If you use tools from Chapter 2 such as induction, inclusion-exclusion, or pigeonholes, please state so explicitly.

For problems 5 through 7, you do **not** need to show your work. You may simply write down your final answer in terms of binomial coefficients, factorials, and numbers. You do **not** need to simplify your answers algebraically. You will be given 1 point for each correct answer, for a total of 15 points.

Problem 1. Find the number of 100-digit positive integers in which no two consecutive digits are the same.

Solution. We select one digit at a time from left to right. There are 9 choices for the first digit, since the first digit should not be a 0. There are 9 choices for each subsequent digit, as it must avoid being the same as the previous digit. This gives 9^{100} such numbers. $\square$

Problem 2. The sequence a_n is given by $a_0 = 3$, $a_1 = 5$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$. Find a formula for a_n for all natural numbers $n \in \mathbb{N}$.

Solution. The associated characteristic polynomial is $x^2-2x-3 = (x-3)(x+1)$, with roots $x = 3$ and $x = -1$. As such, both 3^n and $(-1)^n$ are potential solutions to the recurrence relation. Write $a_n = b \cdot 3^n + c \cdot (-1)^n$. Substituting $n = 0$ and $n = 1$ gives $3 = a_0 = b + c$ and $5 = a_1 = 3b - c$, respectively. Thus $b = 2$ and $c = 1$, yielding the formula $a_n = 2 \cdot 3^n + (-1)^n$. $\square$

Problem 3. Let $n \in \mathbb{N}$ be a natural number. Prove the following identity:

$$\sum_{k=0}^{n} k(n-k)\binom{n}{k} = n(n-1)2^{n-2}.$$

Solution. How many ways can n Girl Scouts be divided into two teams, one to sell cookies in Minneapolis, the other in St. Paul, where each team is led by a Captain?

RHS: Pick the Minneapolis Captain in n ways and then select the St. Paul Captain in $n-1$ ways. (It is highly inappropriate for one to be Captain to both teams!) These two Captains fight over each of the $n-2$ girls. These binary decisions are independent, and thus yielding $n(n-1)2^{n-2}$ outcomes.

LHS: On the other hand, pick k girls for the Minneapolis team, where k ranges from 0 to n. There are $\binom{n}{k}$ ways to do this. These k girls choose one of them to be their Captain. The remaining $n-k$ girls are, of course, the St. Paul team, and choose their own Captain in $n-k$ ways. Since we do not actually care how many girls are in the Minneapolis team, we sum over all possible values of k to get the total number of ways to set up the teams. $\square$

Here is the same solution phrased more formally in the language of sets.

Solution. Count the number of triples (a, b, A) such that $a \in A \subseteq [n]$ and $b \in [n]\backslash A$.

RHS: Let us count the ways to choose a, b, and A, in that order. There are n choices for a. Given such a choice, there are $n-1$ choices for $b \neq a$. Then for each $x \in [n]\backslash\{a, b\}$, we may either include or exclude x from A. These $n-2$ binary decisions are independent, yielding 2^{n-2} choices. In total, there are $n(n-1)2^{n-2}$ such triples.

LHS: On the other hand, let us count the ways to choose A, a, and b, in that order. For each $k \in \{0, 1, \ldots, n\}$, ther are $\binom{n}{k}$ ways to choose A such that $|A| = k$. With A chosen, there are k ways to choose

$a \in A$ and $n-k$ ways to choose $b \notin A$. Since we do not care about the cardinality of A, we sum over k to get all the cases. This yields the left side of the identity, as desired. $\square$

Problem 4. How many ways can you distribute n identical balls amongst g girls and b boys such that *each* girl gets at least as many balls as *all* the boys combined? [You may use summation notation in your answer.]

Solution. Let i be the number of balls we distribute to the boys in total. Give each girl i balls first, and then distribute the remaining $n - gi - i$ balls amongst the girls without restriction. There are $\binom{i+b-1}{b-1}$ ways to distribute i balls to the boys, and $\binom{n-gi-i+g-1}{g-1}$ ways to distribute the remainder to the girls. Summing over all possibilities of i, we get the final answer

$$\sum_{i=0}^{n} \binom{i+b-1}{b-1}\binom{n-gi-i+g-1}{g-1}.$$

$\square$

Problem 5. Suppose there are 200 people. Count the number of different ways to do each of the following:

(a) Form a committee with 5 members.
(b) Form a committee with 5 members total, of whom 2 are designated as "co-chairs."
(c) Partition them into 3 groups of sizes 100, 60, and 40, respectively, where no one is in multiple groups.
(d) Partition them into 4 equal-sized groups, where everyone is in precisely one group.

Solution.

(a): $\binom{200}{5}$.

(b): $\binom{200}{2}\binom{198}{3} = \binom{200}{5}\binom{5}{2}$.

(c): $\binom{200}{100,60,40} = \binom{200}{100}\binom{100}{60}\binom{40}{40} = \frac{200!}{100!60!40!}$.

(d): $\binom{200}{50,50,50,50}/4! = \frac{1}{4!}\binom{200}{50}\binom{150}{50}\binom{100}{50}\binom{50}{50} = \frac{200!}{4!50!50!50!50!}$. The factor of 4! is to account for the fact that the equal-sized groups are not distinguishable from each other. $\square$

Problem 6. Let $n \in \mathbb{N}$ be a natural number and consider a $3n$-by-$3n$ grid. Let X denote the set of shortest (monotonic) paths along the grid lines from $A = (0,0)$ to $B = (3n,3n)$. Find the number of such paths given certain restrictions. [Your answer should be in terms of n, and can have multiple binomial coefficients.]

(a) No restrictions; in other words, find $|X|$.
(b) Visit $C = (n,n)$ and $E = (2n,2n)$.
(c) Visit C or E (or both).
(d) Visit $F = (n,2n)$ or $D = (2n,n)$.
(e) Avoid C, D, E, and F.

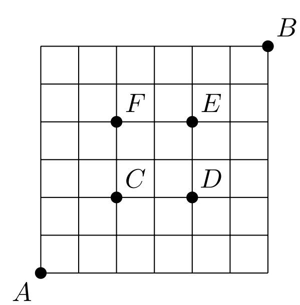


FIGURE 6. The case for $n = 2$.

Solution.

(a): $\binom{6n}{3n}$.

(b): $\binom{2n}{n}^3$.

(c): $2\binom{2n}{n}\binom{4n}{2n} - \binom{2n}{n}^3$, by principle of inclusion-exclusion.

(d): $2\binom{3n}{n}^2$; note that it is impossible to visit both F and D.

(e): $\binom{6n}{3n} - 2\binom{2n}{n}\binom{4n}{2n} - 2\binom{3n}{n}^2 + \binom{2n}{n}^3 + 4\binom{2n}{n}\binom{3n}{n} - 2\binom{2n}{n}^2$, by principle of inclusion-exclusion. □

Problem 7. Suppose you have a small garden divided into a 2-by-3 grid (see figure below). You have 6 flowers and plan to plant each one in a different square in the grid. Count the number of different floral arrangements under the following circumstances:

(a) You have 6 different kinds of flowers.
(b) You have 4 chrysanthemums that you consider the same, 1 rose, and 1 dandelion.
(c) You have 4 chrysanthemums that you consider the same and 2 roses that you consider the same.

Now suppose you are putting 6 cookies in a 2-by-3 box, one in each compartment (see the same figure below). The difference between a box and a garden is that you can't tell if a box is turned around. [Do not flip the box upside-down, for the cookies will fall out and make a mess.] Count the number of different snack arrangements under the following circumstances:

(d) You have 6 different kinds of cookies.
(e) You have 4 chocolate-chip cookies that you consider the same, 1 raisin cookie, and 1 dandelion-flavoured cookie.
(f) You have 4 chocolate-chip cookies that you consider the same and 2 raisin cookies that you consider the same.

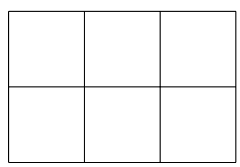


FIGURE 7. A garden divided into a 2-by-3 grid. Incidentally, also a 2-by-3 box with 6 compartments.

Solution.

(a): $6! = 720$.

(b): $6 \cdot 5 = 30$. Plant the rose, then plant the dandelion.

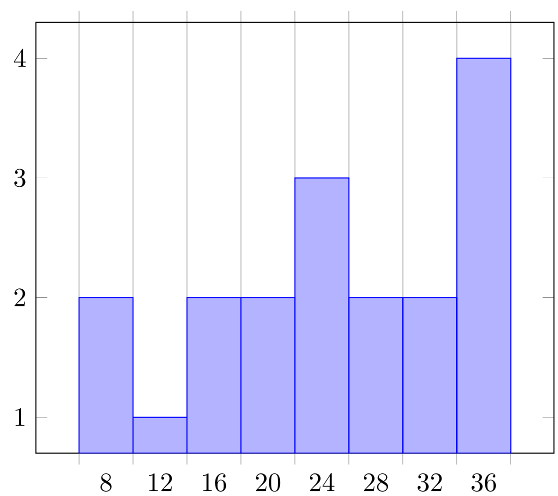
(c): $\binom{6}{2} = 15$. Plant the roses first.

(d): $6!/2 = 360$. Divide by 2 to account for rotation.

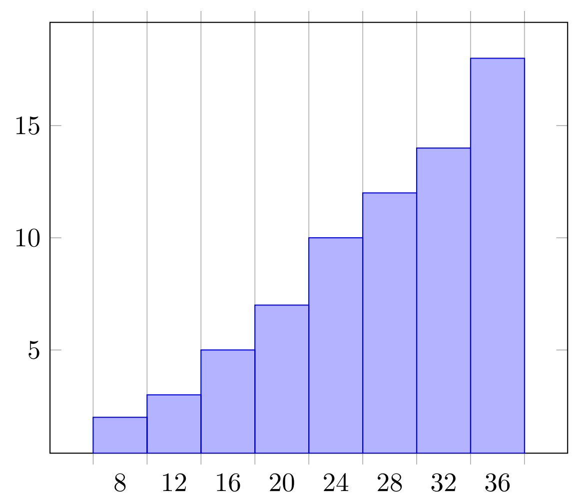
(e): $3 \cdot 5 = 15$. Put raisin in the top row, then put dandelion anywhere else.

(f): $3 + (15-3)/2 = 9$. Three of the 15 floral arrangements from part (c) have rotational symmetry. The others form 6 pairs of floral arrangements that are indistinguishable as snack arrangements. □

Problem	Mean	Stdev
Problem 1 (7 points)	5.28	2.76
Problem 2 (7 points)	5.67	2.35
Problem 3 (7 points)	2.44	2.68
Problem 4 (7 points)	2.61	2.91
Problem 5 (4 points)	2.56	1.00
Problem 6 (5 points)	3.06	1.21
Problem 7 (6 points)	4.06	1.55
$\sum$ (43 points total)	23.10	12.39



(A) Histogram of scores.



(B) Cumulative histogram of scores.