1. LIMITS AND CONTINUITY

1.1. Definitions. Let \( f \) be a function of two variables whose domain \( D \) includes points arbitrarily close to \((a, b)\). Then the limit \( \lim_{(x,y) \to (a,b)} f(x, y) = L \) exists if for every number \( \varepsilon > 0 \) there is a corresponding number \( \delta > 0 \) such that if \((x, y) \in D\), and \( 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \) then \( |f(x, y) - L| < \varepsilon \).

It is continuous at \((a, b)\) if \( \lim_{(x,y) \to (a,b)} f(x, y) = f(a, b) \).

1.2. Exercise 15.2.5. Find the limit, if it exists, or show that the limit does not exist.

\[ \lim_{(x,y) \to (1,2)} (5x^3 - x^2y^2). \]

Solution. Recall that polynomials are continuous, hence we may employ the substitution method to get the limit: \( 5 \cdot 1^3 - 1^2 \cdot 2^2 = 1. \) \( \square \)

1.3. Exercise 15.2.16. Find the limit, if it exists, or show that the limit does not exist.

\[ \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}. \]

Solution. Notice that \( 0 \leq \frac{x^2}{x^2 + 2y^2} \leq 1 \), so the quantity above is squeezed between \( 0 \) and \( \sin^2 y \to 0 \), hence the limit is \( 0. \) \( \square \)

1.4. Exercise 15.2.38. Determine the set of points at which the function is continuous:

\[ f(x, y) = \begin{cases} 
\frac{x^2}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{otherwise}.
\end{cases} \]

Solution. Notice that \( x^2 + xy + y^2 = 0 \) only at \((x, y) = (0, 0)\). Indeed, the minimum with respect to \( x \) is when \( x = -y/2 \), which makes the expression \( 3y^2/4 \). Therefore the function is continuous at all points besides the origin.

As for the origin, it is not continuous there. Indeed, along the line \( y = x \) the limit is \( \lim_{x \to 0} \frac{x^2}{x^2 + 2x^2 + 2x^2} = \frac{1}{2} \). Whereas along the line \( y = 2x \) the limit is \( \lim_{x \to 0} \frac{2x^2}{x^2 + 2x^2 + 4x^2} = \frac{2}{7}. \) \( \square \)

1.5. Exercise 15.2.43. Discuss the continuity of the function

\[ f(x, y) = \begin{cases} 
\frac{\sin xy}{xy} & \text{if } xy \neq 0, \\
1 & \text{otherwise}.
\end{cases} \]
Solution. Obviously $f$ is continuous at all $xy \neq 0$. Now fix a point $(x_0, y_0)$ such that $x_0 y_0 = 0$. Take an arbitrary continuous path $C(t) = (x(t), y(t))$ such that $C(0) = (x(0), y(0)) = (x_0, y_0)$, and let $z(t) = x(t)y(t)$. If there is a small neighbourhood around 0 such that the path lies in the set $xy = 0$, then the limit is 1. Otherwise, we may assume $f(C(t)) = \sin(x(t)y(t))/(x(t)y(t)) = \sin(z(t))/z(t)$. Then we seek the limit $\lim_{t \to 0} f(x(t), y(t)) = \lim_{t \to 0} \sin(z(t))/z(t))$. But as $t \to 0$, we have $z(t) \to z(0) = x(0)y(0) = 0$, so the limit is 1 as $\lim_{z \to 0} \sin z/z = 1$. Hence we conclude that $f$ is continuous everywhere.

Alternatively, consider $g(x, y) = x y$ and $h(t) = \sin t/t$ for $t \neq 0$ and $h(t) = 1$ for $t = 0$. Then $f(x, y) = h(g(x, y))$, and both $h$ and $g$ are continuous everywhere, hence so is $f$.

\[\square\]

1.6. Exercise 15.2.45. Show that the function $f$ given by $f(x) = |x|$ is continuous on $\mathbb{R}^n$.

Solution. First recall that $a \cdot x = |a||x|\cos \theta$, where $\theta$ is the angle between them. Therefore $a \cdot x \leq |a||x|$. Now compute $|f(x) - f(a)| = ||x| - |a|| = \sqrt{|x - a|^2} = \sqrt{|x|^2 - 2|x||a| + |a|^2} \leq \sqrt{x \cdot x - 2x \cdot a + a \cdot a} = \sqrt{(x - a) \cdot (x - a)} = |x - a|$. So if we make $a$ approach $x$, namely, $|x - a| \to 0$, then $|f(x) - f(a)| \to 0$, namely, $f(x)$ approaches $f(a)$, as desired.

\[\square\]

2. Partial Derivatives

2.1. Definitions. The partial derivative of $f$ with respect to $x$ at $(a, b)$ is $f_x(a, b) = g'(a)$ where $g(x) = f(x, b)$.

2.2. Exercise 15.3.16. Find the first partial derivatives of $f(x, y) = x^4y^3 + 8x^2y$.

Solution. Treating $y$ constant we get $f_x(x, y) = 4x^3y^3 + 16xy$. Treating $x$ constant we get $f_y(x, y) = 3x^4y^2 + 8x^2$.

\[\square\]

2.3. Exercise 15.3.28. Find the first partial derivatives of

$$f(x, y) = \int_y^x \cos(t^2) \, dt.$$ 

Solution. Let $A(x) = \int_y^x \cos(t^2) \, dt$, by FTC, we get $A'(x) = \cos(x^2)$. So $f(x, y) = A(x) - A(y)$, giving $f_x(x, y) = A'(x) = \cos(x^2)$ and $f_y(x, y) = -A'(y) = -\cos(y^2)$.

\[\square\]

2.4. Exercise 15.3.45. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, given

$$x^2 + y^2 + z^2 = 3xyz.$$ 

Solution. Applying $\frac{\partial}{\partial x}$ we get $2x + 2z \frac{\partial z}{\partial x} = 3yz + 3xy \frac{\partial z}{\partial x}$, solving, we get $\frac{\partial z}{\partial x} = \frac{3yz - 2x}{2x - 3xy}$. Similarly, $\frac{\partial z}{\partial y} = \frac{3xz - 2y}{2x - 3xy}$.

\[\square\]

2.5. Exercise 15.3.94. If $f(x, y) = \sqrt{x^2 + y^2}$, find $f_x(0, 0)$.

Solution. Let $g(x) = f(x, 0) = x$, so $g'(x) = 1$. Now $f_x(0, 0) = g'(0) = 1$. Notice that it would be much more difficult to calculate $f_x(x, y)$ directly with $y$ an arbitrary constant.

\[\square\]