MATH 31A DISCUSSION

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MORE APPLICATIONS OF THE DERIVATIVE

1. MVT AND MONOTONICITY

1.1. Mean Value Theorem. Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ 

In particular, if $f(a) = f(b)$, we get Rolle’s Theorem.

1.2. Exercise 4.3.42. Show that $f(x) = x^3 - 2x^2 + 2x$ is an increasing function.

Solution. Notice $f'(x) = 3x^2 - 4x + 2$. What is its minimum? Find its critical points: $f''(x) = 6x - 4$, so $x = \frac{2}{3}$ is the critical point. So $f'(x)$ has its minimum at $x = \frac{2}{3}$, which is $f'(\frac{2}{3}) = \frac{2}{3}$. So $f'(x) > 0$, thus $f(x)$ is increasing. □

1.3. Exercise 4.3.53–55. Prove that if $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for $x \geq 0$, then $f(x) \leq g(x)$ for all $x \geq 0$. Prove the following:

(a) $\sin x \leq x$ for $x \geq 0$.
(b) $\cos x \geq 1 - \frac{1}{2}x^2$.
(c) $\sin x \geq x - \frac{1}{6}x^3$.
(d) $\cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Solution. Let $h(x) = f(x) - g(x)$. Notice $h'(x) = f'(x) - g'(x) \leq 0$. So $h(x)$ is non-increasing. Since $h(0) = 0$, we have that for $x \geq 0$, $h(0) \leq 0$. So $f(x) - g(x) \leq 0$, thus $f(x) \leq g(x)$, as desired.

Since $\sin x$ and $x$ agree at $x = 0$, and the derivatives $\cos x \leq 1$ as required, we apply what we got above to get the desired result. The rest follows similarly. □

2. GRAPHS

2.1. Basics.

2.1.1. Concavity. If $f'(x)$ is increasing (or $f''(x) > 0$), then $f$ is concave up at $x$. If $f'(x)$ is decreasing (or $f''(x) < 0$), then $f$ is concave down at $x$.

2.1.2. Inflection. If $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$, then $f(x)$ has a point of inflection at $x = c$.

2.1.3. Second Derivative Test. Let $f$ be differentiable and $c$ a critical point.

(a) If $f''(c) > 0$ then $f(c)$ is a local minimum.
(b) If $f''(c) < 0$ then $f(c)$ is a local maximum.
(c) If $f''(c) = 0$ then it is inconclusive, $f(c)$ may be a local min, max, or neither.
2.2. **Exercise 4.4.24.** Find the critical points of \( f(x) = \sin^2 x + \cos x \), \( x \) in \([0, \pi]\), and use the Second Derivative Test to determine whether each corresponds to a local minimum or maximum.

**Solution.** Notice \( f'(x) = 2 \sin x \cos x - \sin x \). Setting \( f'(x) = 0 \) and solving, we get \( \sin x (2 \cos x - 1) = 0 \), so \( x = 0, \pi/3, \pi \). Now \( f''(x) = 2 \cos^2 x - 2 \sin^2 x - \cos x \). So \( f''(0) = 1, f''(\pi/3) = -3/2, f''(\pi) = 3 \), yielding local minima at \( x = 0, \pi \) and maximum at \( x = \pi/3 \). \( \square \)

2.3. **Exercise 4.4.53.** If \( f'(c) = 0 \) and \( f(c) \) is neither a local min or max, must \( x = c \) be a point of inflection? This is true of most “reasonable” examples, but it is not true in general. Let

\[
f(x) = \begin{cases} 
  x^2 \sin \frac{1}{x} & \text{for } x \neq 0, \\
  0 & \text{for } x = 0.
\end{cases}
\]

- Use the limit definition of the derivative to show that \( f'(0) \) exists and \( f'(0) = 0 \).
- Show that \( f(0) \) is neither a local min nor max.
- Show that \( f'(x) \) changes sign infinitely often near \( x = 0 \) and conclude that \( f(x) \) does not have a point of inflection at \( x = 0 \).

**Solution.** Recall Exercise 3.7.92 from 10/20.

Recall \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \). So by definition, \( f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{x \to 0} 2h \sin \frac{1}{h} \). Using Squeeze Theorem and \(-|h| \leq h \sin \frac{1}{h} \leq |h|\), we get that \( f'(0) = 0 \). Away from \( x = 0 \), we can use the formula and get \( f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} (-1) \frac{1}{x^2} = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \). Now \( \lim_{x \to 0} f'(x) \) does not exist since \( \lim_{x \to 0} 2x \sin \frac{1}{x} = 0 \) by Squeeze Theorem but \( \lim_{x \to 0} \cos \frac{1}{x} \) does not exist. \( \square \)

2.4. **Bonus Question.** Assume \( f''(x) \) exists and \( f''(x) > 0 \) for all \( x \). Show that \( f(x) \) cannot be always negative.

**Solution.** If \( f'(x) \equiv 0 \), then \( f''(x) \equiv 0 \), a contradiction. So there exists \( b \) such that \( f'(b) \neq 0 \). Consider the tangent line at \( x = b \) to \( f(x) \). It is given by the equation \( y = f'(b)(x - b) + f(b) \). Consider \( g(x) = f(x) - f'(b)(x - b) - f(b) \). Notice that \( g'(x) = f'(x) - f'(b) \), and \( g''(x) = f''(x) \). So \( g(b) = g'(b) = 0 \), and \( g''(x) > 0 \) for all \( x \). Hence \( g'(x) \) is increasing. In particular, \( g'(x) < 0 \) for \( x < b \), and \( g'(x) > 0 \) for \( x > b \). If \( x > b \), then by MVT, we get

\[
\frac{g(x) - g(b)}{x - b} = g'(c)
\]

for some \( c \) in the interval \((b, x)\). In otherwords, since \( c > b \), we have that \( g'(c) > 0 \) and \( x - b > 0 \), hence \( g(x) - g(b) > 0 \). Similarly, if \( x < b \), we get \( g'(c) < 0 \), \( x - b < 0 \), so \( g(x) - g(b) > 0 \) as well. We thus conclude \( g(x) \geq g(b) \) for all \( x \). So \( f(x) \geq f'(b)(x - b) + f(b) \) for all \( x \). Since \( f'(b) \neq 0 \), there exists \( x \) far enough from the origin such that \( f'(b)(x - b) + f(b) > 0 \), as desired. \( \square \)