Applications of the Derivative

1. Related Rates

1.1. Exercise 3.9.44. A wheel of radius \( r \) is centred at the origin. As it rotates, the rod of length \( L \) attached at the point \( P \) drives a piston back and forth in a straight line. Let \( x \) be the distance from the origin to the point \( Q \) at the end of the rod.

(a) Use the Pythagorean Theorem to show that
\[
L^2 = (x - r \cos \theta)^2 + r^2 \sin^2 \theta.
\]

(b) Differentiate part (a) with respect to \( t \) to prove that
\[
2(x - r \cos \theta) \left( \frac{dx}{dt} + r \sin \theta \frac{d\theta}{dt} \right) + 2r^2 \sin \theta \cos \theta \frac{d\theta}{dt} = 0.
\]

(c) Calculate the speed of the piston when \( \theta = \frac{\pi}{2} \), assuming that \( r = 10 \text{ cm} \), \( L = 30 \text{ cm} \), and the wheel rotates at 4 revolutions per minute.

**Solution.** Parts (a) and (b) are straightforward. 4 revolutions per minute means \( \frac{d\theta}{dt} = 4 \cdot 2\pi \) per minute. From part (a), we get \( 30^2 = x^2 + 10^2 \), so \( x = 20\sqrt{2} \). Plugging in, we get
\[
2(20\sqrt{2} - 0) \left( \frac{dx}{dt} + 10 \cdot 8\pi \right) + 0 = 0.
\]
So \( \frac{dx}{dt} = -80\pi \text{ cm per minute}. \)

2. Linear Approximations

2.1. Approximating Change. If \( f \) is differentiable at \( x = a \) and \( \Delta x \) is small, then
\[
\Delta f \approx f'(a) \Delta x
\]
where \( \Delta f = f(a + \Delta x) - f(a) \).

2.2. Linearisation. If \( f \) is differentiable at \( x = a \), and \( x \) is close to \( a \), then
\[
f(x) \approx L(x) = f'(a)(x - a) + f(a).
\]

3. Extrema


3.1.1. Critical Points. A number \( c \) in the domain of \( f \) is called a critical point if either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

3.1.2. Local Extrema. If \( f(c) \) is a local extremum, then \( c \) is a critical point of \( f \).

3.1.3. Extrema on Closed Interval. If \( f(x) \) is continuous on \( [a, b] \), and \( f(c) \) be an extremum on \( [a, b] \). Then \( c \) is either a critical point or one of the endpoints \( a \) or \( b \).
3.1.4. *Rolle’s Theorem.* Assume that \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f(a) = f(b) \), then there exists a number \( c \) between \( a \) and \( b \) such that \( f'(c) = 0 \).

3.2. **Exercise 4.2.39.** Find the maximum and minimum values of \( y = \sin x \cos x \) on \([0, \frac{\pi}{2}]\).

*Solution.* Notice \( y' = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 \). If \( y' = 0 \), then \( \cos x = \pm \sqrt{\frac{1}{2}} \). In \([0, \frac{\pi}{2}]\), this occur at \( x = \frac{\pi}{4} \). Now \( f(0) = f(\frac{\pi}{2}) = 0 \), and \( f(\frac{\pi}{4}) = \frac{1}{2} \). So min is 0 and max is \( \frac{1}{2} \).

3.3. **Exercise 4.2.73–74.** Show that \( f(x) = x^2 - 2x + 3 \) takes on only positive values. Find conditions on \( r \) and \( s \) under which the quadratic function \( f(x) = x^2 + rx + s \) takes on only positive values. Show that if \( f \) takes on both positive and negative values, then its minimum value occurs at the midpoint between the two roots.

*Solution.* For \( f(x) = x^2 - 2x + 3 \), we have \( f'(x) = 2x - 2 \), so \( x = 1 \) is critical point, and \( f(1) = 2 > 0 \). More generally, for \( f(x) = x^2 + rx + s \), we have \( f'(x) = 2x + r \), so \( x = -\frac{r}{2} \) is critical point. Now \( f(-\frac{r}{2}) = s - \frac{r^2}{4} \). So if we want this to be positive, we must have \( s > \frac{r^2}{4} \). If \( f \) takes on both positive and negative values, then the roots are \( x = -\frac{r \pm \sqrt{r^2 - 4s}}{2} \), whose midpoint is \( x = -\frac{r}{2} \), as desired.