1. Derivatives

1.1. Basics. Given a function \( f(x) \). The slope of the tangent line at \( x = c \) is \( f'(c) \).

1.1.1. Power Rule. For all exponents \( n \in \mathbb{R} \), \( \frac{d}{dx} x^n = nx^{n-1} \). Not for \( e^x \), \( x^x \).

1.1.2. Linearity Rules. If \( f \) and \( g \) are differentiable functions, \( c \in \mathbb{R} \), then \( cf \) and \( f + g \) are differentiable. Indeed, \( (f + g)' = f' + g' \) and \( (cf)' = cf' \).

1.1.3. Product and Quotient Rules. If \( f \) and \( g \) are differentiable, \( (fg)' = fg' + gf' \). And \( (f/g)' = (f'g - gf')/g^2 \).

1.2. Exercise 3.2.46. Sketch the graphs of \( f(x) = x^2 - 5x + 4 \) and \( g(x) = -2x + 3 \). Find the value of \( x \) at which the graphs have parallel tangent lines.

Solution. We need \( f'(x) = g'(x) \). Notice \( f'(x) = 2x - 5 \) and \( g'(x) = -2 \). So we solve \( 2x - 5 = -2 \) to get \( x = \frac{3}{2} \).

1.3. Exercise 3.2.52. Show that if the tangent lines to the graph of \( y = \frac{1}{3}x^3 - x^2 \) at \( x = a \) and \( x = b \) are parallel, then either \( a = b \) or \( a + b = 2 \).

Solution. We want \( y'(a) = y'(b) \). Now \( y' = x^2 - 2x \). So if \( a^2 - 2a = b^2 - 2b \), then \( (a^2 - b^2) = 2(a - b) \), giving \( (a + b)(a - b) = 2(a - b) \).

1.4. Exercise 3.3.55. Let \( f(x) \) be a polynomial. Then \( c \) is a root of \( f(x) \) if and only if \( f(x) = (x - c)g(x) \) for some polynomial \( g(x) \). We say that \( c \) is a multiple root if \( f(x) = (x - c)^2h(x) \) for some polynomial \( h(x) \).

Show that \( c \) is a multiple root of \( f(x) \) if and only if \( c \) is a root of both \( f(x) \) and \( f'(x) \).

Solution. Suppose \( c \) is a multiple root of \( f(x) \). Then there exists some polynomial \( h(x) \) such that \( f(x) = (x - c)^2h(x) \). Obviously \( c \) is a root of \( f(x) \). Let \( g(x) = (x - c)h(x) \), thus \( f(x) = (x - c)g(x) \). Using the Product Rule, we have \( f'(x) = g(x) + (x - c)g'(x) \). Notice \( f'(c) = g(c) = 0 \), so \( c \) is a root of \( f'(x) \), as desired.

Conversely, suppose \( c \) is a root of both \( f(x) \) and \( f'(x) \). As \( c \) is a root of \( f(x) \), there exists some polynomial \( g(x) \) such that \( f(x) = (x - c)g(x) \). By the Product Rule, we again have \( f'(x) = g(x) + (x - c)g'(x) \) and \( f'(c) = g(c) \). Since \( c \) is a root of \( f'(x) \), we have \( f'(c) = 0 \), hence \( g(c) = 0 \). We conclude that \( g(x) = (x - c)h(x) \) for some polynomial \( h(x) \), hence \( f(x) = (x - c)^2h(x) \), as desired.

1.5. Exercise 3.3.56. Use Exercise 55 to determine whether \( c = -1 \) is a multiple root of the polynomial \( f(x) = x^4 + x^3 - 5x^2 - 3x + 2 \).

Solution. First check \( f(-1) = 0 \), so \(-1\) is a root of \( f \). Now \( f'(x) = 4x^3 + 3x^2 - 10x - 3 \). So \( f'(-1) = 6 \neq 0 \), so \(-1\) is not a multiple root.
1.6. Exercise 3.4.32. It takes a stone 3 s to hit the ground when dropped from the top of a building. How high is the building and what is the stone’s velocity upon impact.

Solution. The position is \( s(t) = s_0 + v_0 t - \frac{1}{2} g t^2 \), where \( s_0 \) is the initial height, \( v_0 \) is the initial velocity, and \( g \approx 9.8 \text{ m/s}^2 \). We have \( v_0 = 0 \), and \( s(3) = 0 \). So we get \( s(3) = s_0 - \frac{1}{2} g (3)^2 = 0 \). So the initial height is \( 4.5g \approx 44.1 \text{ m} \).

The velocity is \( v(t) = v_0 - gt \). So \( v(3) = -3g \approx -29.4 \text{ m/s} \).  

1.7. Exercise 3.4.33. A ball is tossed up vertically from ground level and returns to earth 4 s later. What was the initial velocity of the stone and how high did it go?

Solution. We have \( s_0 = 0 \), \( s(4) = 0 \), so we can solve for \( v_0 \). Indeed, \( s(4) = 0 + 4v_0 - 8g = 0 \), so \( v_0 = 2g = 19.6 \text{ m/s} \).

The maximum height occurs when the derivative is zero. So \( s'(t) = v(t) = v_0 - gt = 0 \) gives \( t = v_0 / g = 2 \). This confirms what we thought this is the half way point. The height is \( s(2) = 0 + 2v_0 - 2g = 2g = 19.6 \text{ m} \).