1. Introduction

Lecture 1
- Instructor: Steve Butler.
- Location: HAINES 39.

Sections 1A and 1B
- Email: mailto:jedyang@ucla.edu
- Office: MS 6617A.
- Discussion Location: MS 5117 (T) and 5138 (R).
- Website: http://www.math.ucla.edu/~jedyang/31a.1.10w/
- SMC: Jan. 6–Mar. 11, M–R 09:00–15:00, MS 3974, T 12:00–13:00.

2. Administration
- HW due Fridays in lecture, can turn in early to me, and I will hand back in section.
- Confirm office hour.

3. Precalculus Review

3.1. Exercise 1.2.21. Find the equation of the perpendicular bisector of the segment joining (1, 2) and (5, 4).

Solution. Slope of segment is $m_1 = \frac{4 - 2}{5 - 1} = \frac{1}{2}$. Slope of perpendicular bisector is $m_2 = -1/m_1 = -2$. Mid point is $(\frac{1+5}{2}, \frac{2+4}{2})$. So the equation can be written as $y - 3 = -2(x - 3)$. □

3.2. Exercise 1.2.23. Find the equation of the line with $x$-intercept $x = 4$ and $y$-intercept $y = 3$.

Solution. Equation of the line is $y = mx + b$, where $b$ is the $y$-intercept, hence $b = 3$. The $x$-intercept $x = 4$ will yield $y = 0$ (by definition), so substituting, we may solve for $m$. We get $0 = 4m + 3$, hence $m = -\frac{3}{4}$. So the equation can be written as $y = -\frac{3}{4}x + 3$. □

3.3. Exercise 1.2.24. A line of slope $m = 2$ passes through $(1, 4)$. Find $y$ such that $(3, y)$ lies on the line.

Solution. Equation of the line is $y = mx + b$, substituting $m = 2$ and $(1, 4)$, we get $4 = 2(1) + b$, hence $b = 2$. So the line is $y = 2x + 2$. Substituting $(3, y)$ gives $y = 2(3) + 2$, hence $y = 8$. □
Solution. One way is to write down an equation of the line in point-slope form: $y = 2(x - 1) + 4$. Then we see clearly that if $x = 3$, then $y = 8$. Alternatively, the slope $m$ is the change of $y$ over the change of $x$. Symbolically, $m = \frac{\Delta y}{\Delta x}$, or $\Delta y = m \Delta x$. This concept will be useful later when we deal with differentials $dy = m \, dx$. Since the change in $x$ is $\Delta x = 3 - 1 = 2$, we get that the change in $y$ is $\Delta y = y - 4 = 2 \cdot 2 = 4$, hence $y = 8$. This method seems longer, but conceptually it is easier to do in one’s head, and will lead to intuition for calculus later. □

3.4. Exercise 1.4.55. Use the addition formulae for sine and cosine to prove

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \quad (1)$$
$$\cot(a - b) = \frac{\cot a \cot b + 1}{\cot b - \cot a} \quad (2)$$

Proof. Recall that

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \quad (3)$$
$$\cos(a + b) = \cos a \cos b - \sin a \sin b \quad (4)$$

Now

$$\tan(a + b) = \frac{\sin(a + b)}{\cos(a + b)} \quad (5)$$
$$= \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b} \quad (6)$$
$$= \frac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a \sin b}{\cos a \cos b}} \quad (7)$$
$$= \frac{\tan a + \tan b}{1 - \tan a \tan b} \quad (8)$$

where we get from (6) to (7) by dividing top and bottom by $\cos a \cos b$.

The case for cotangent is completely analogous. Remember $\cot x = \frac{\cos x}{\sin x}$ and that $\sin(-b) = -\sin(b)$ and $\cos(-b) = \cos(b)$. Work out the details and convince yourself. □

3.5. Exercise 1.4.56. Let $\theta$ be the angle between the line $y = mx + b$ and the $x$-axis. Prove that $m = \tan \theta$.

Proof. This is trivial. □

3.6. Exercise 1.4.57. Let $L_1$ and $L_2$ be the lines of slope $m_1$ and $m_2$, respectively. Show that the angle $\theta$ between $L_1$ and $L_2$ satisfies $\cot \theta = \frac{m_2 m_1 + 1}{m_2 - m_1}$.

Proof. This is immediate by using Exercises 55 and 56. □

3.7. Exercise 1.4.58. Perpendicular Lines. Use Exercise 57 to prove that two lines with nonzero slopes $m_1$ and $m_2$ are perpendicular if and only if $m_2 = -1/m_1$.

Proof. What is $\cot(\pi/2)$? □
3.8. Exercise 1.4.59. Apply the double-angle formula to prove:

(a) $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$.
(b) $\cos \frac{\pi}{12} = \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}}$.

Guess the values of $\cos \frac{\pi}{32}$ and of $\cos \frac{\pi}{16}$ for all $n$.

Proof. Recall $\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$. For the general case, let $a_0 = 0$ and define inductively $a_n = \sqrt{2 + a_{n-1}}$. We claim that for $n \geq 1$, we have $\cos \frac{\pi}{2^n} = \frac{1}{2} a_{n-1}$. The base case is trivial. By induction, assume $\cos \frac{\pi}{2^n} = \frac{1}{2} a_{n-1}$. By the half-angle formula, we get $\cos \frac{\pi}{2^{n+1}} = \sqrt{\frac{1}{2}(1 + \frac{1}{2}a_{n-1})} = \sqrt{\frac{1}{4}(2 + a_{n-1})} = \frac{1}{2} a_n$. □

4. Basic Limit Laws

4.1. Basic Limit Laws. Assume that $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist. Then:

(a) Sum Law:
$$\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x).$$
(b) Constant Multiple Law: For any number $k \in \mathbb{R}$,
$$\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x).$$
(c) Product Law:
$$\lim_{x \to c} (f(x)g(x)) = \left(\lim_{x \to c} f(x)\right) \left(\lim_{x \to c} g(x)\right).$$
(d) Quotient Law: If $\lim_{x \to c} g(x) \neq 0$, then
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}.$$

4.2. Exercise 2.3.22. Evaluate the limit $\lim_{z \to 1} \frac{z^2 - 1}{z + 1}$.

Solution. Recall that $\lim_{z \to 1} z = 1$ and $\lim_{z \to 1} 1 = 1$. By the Quotient Law, $\lim_{z \to 1} \frac{z^2 - 1}{z + 1} = \frac{\lim_{z \to 1} (z^2 - 1)}{\lim_{z \to 1} (z + 1)} = \frac{1}{2} = 1$. By the Sum Law, $\lim_{z \to 1} z - 1 + z = \lim_{z \to 1} z - 1 + \lim_{z \to 1} z = 1 + 1 = 2$. By the Sum Law, $\lim_{z \to 1} z + 1 = 2$. So by the Quotient Law, $\lim_{z \to 1} \frac{z^2 - 1}{z + 1} = \frac{\lim_{z \to 1} (z^2 - 1)}{\lim_{z \to 1} (z + 1)} = \frac{2}{2} = 1$. □

4.3. Exercise 2.3.29. Can the Quotient Law be applied to evaluate $\lim_{x \to 0} \frac{\sin x}{x}$?

Solution. The Quotient Law requires the limit of the denominator, namely, $\lim_{x \to 0} x$, to exist and be nonzero. This is not the case, so we cannot apply directly. □

4.4. Exercise 2.3.30. Show that the Product Law cannot be used to evaluate $\lim_{x \to \pi/2} (x - \pi/2) \tan x$.

Solution. The Product Law requires the limit of each factor to exist. However, $\lim_{x \to \pi/2} \tan x$ does not exist. □

4.5. Exercise 2.3.31. Give an example where $\lim_{x \to 0} (f(x) + g(x))$ exists but neither $\lim_{x \to 0} f(x)$ nor $\lim_{x \to 0} g(x)$ exists.
Solution. Let \( f(x) \) be any function defined on a neighborhood of 0 (but not necessarily at 0) such that \( \lim_{x \to 0} f(x) \) does not exist (e.g., \( f(x) = 1/x \)). Let \( g(x) = -f(x) \). Then of course \( \lim_{x \to 0} g(x) \) also does not exist (otherwise by the Constant Multiple Law, \( \lim_{x \to 0} f(x) \) also exists). But notice \( f(x) + g(x) \) is identically zero in a neighborhood of 0 (but not necessarily at 0). So \( \lim_{x \to 0} (f(x) + g(x)) = 0 \) exists.

4.6. Exercise 2.3.32. Assume that the limit \( L_a = \lim_{x \to 0} a^x - 1 \) exists and that \( \lim_{x \to 0} a^x = 1 \) for all \( a > 0 \). Prove that \( L_{ab} = L_a + L_b \) for \( a, b > 0 \). [Hint: \( (ab)^x - 1 = a^x (b^x - 1) + (a^x - 1) \).]

Solution. By definition, \( L_{ab} = \lim_{x \to 0} (ab)^x - 1 = \lim_{x \to 0} a^x b^x - 1 + a^x - 1 \). Since \( \lim_{x \to 0} a^x = 1 \) by assumption and \( \lim_{x \to 0} b^x - 1 = L_b \) exists by assumption, the Product Law states \( \lim_{x \to 0} a^x b^x - 1 = L_b \). Now \( \lim_{x \to 0} \frac{a^x - 1}{x} = L_a \) by assumption, so the Sum Law yields \( \lim_{x \to 0} a^x b^x - 1 + \frac{a^x - 1}{x} = L_b + L_a \).

4.7. Exercise 2.3.38. Assuming that \( \lim_{x \to 0} \frac{f(x)}{x} = 1 \), which of the following statements is necessarily true?

(a) \( f(0) = 0 \).
(b) \( \lim_{x \to 0} f(x) = 0 \).

Solution. Remember that the value of \( f(x) \) at \( x = 0 \) never matters when we evaluate the limit \( \lim_{x \to 0} f(x) \). So (a) is not (necessarily) true.

Recall that \( \lim_{x \to 0} x = 0 \), so by the Product Law, \( \lim_{x \to 0} f(x) = \lim_{x \to 0} x \cdot \frac{f(x)}{x} = \lim_{x \to 0} x \cdot \lim_{x \to 0} \frac{f(x)}{x} = 0 \cdot 1 = 0 \). Since \( \lim_{x \to 0} \frac{f(x)}{x} = 1 \), and \( \lim_{x \to 0} x = 0 \), we get \( \lim_{x \to 0} f(x) = 0 \).