Math 31A Discussion

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1. Theorems

1.1. Area Between Graphs. If \( f(x) \geq g(x) \) on \([a, b]\), then the area between the graphs is \( \int_{a}^{b} f(x) - g(x) \, dx \).

1.2. Volume of a Solid. If the cross section of a solid between \([a, b]\) has area \( A(x) \), then the volume is \( \int_{a}^{b} A(x) \, dx \).

1.3. Solid of Revolution. If a solid of revolution is formed with radii \( r < R \) on \([a, b]\), then the volume is \( \pi \int_{a}^{b} (R^2 - r^2) \, dx \).

2. Applications of the Integral

2.1. Exercise 6.1.47. Set up (but do not evaluate) an integral that expresses the area between the circles \( x^2 + y^2 = 2 \) and \( x^2 + (y - 1)^2 = 1 \).

Solution. Solving \( x^2 + y^2 = 2 \) and \( x^2 + (y - 1)^2 = 1 \), we get \( y = 1 \) and \( x = \pm 1 \). So the region is between \( x = -1 \) and \( x = 1 \), above \( y = 1 - \sqrt{1 - x^2} \) and below \( y = \sqrt{2 - x^2} \). Thus an integral representing the region is \( \int_{-1}^{1} \sqrt{2 - x^2} - (1 - \sqrt{1 - x^2}) \, dx \). (The area is \( \pi - 1 \).) □

2.2. Exercise 6.2.21. Let \( S \) be the solid obtained by intersecting two cylinders of radius \( r \) whose axes are perpendicular. Find the volume of \( S \) as a function of \( r \).

Solution. The horizontal cross section of each cylinder at distance \( y \) from the central axis is a rectangular strip. The strip’s width \( w \) satisfies the Pythagorean relationship \( (w/2)^2 + y^2 = r^2 \), hence \( w = 2\sqrt{r^2 - y^2} \).

The area of the horizontal cross section of \( S \) at distance \( y \) is thus \( A = w^2 = 4(r^2 - y^2) \).

Finally, the volume of \( S \) is thus \( V = \int_{-r}^{r} 4(r^2 - y^2) \, dy = 4 \left[ r^2 y - y^3/3 \right]_{-r}^{r} = 8(r^3 - r^3/3) = \frac{16}{3} r^3 \). □

2.3. Exercise 6.3.36. Find volume of the solid obtained by rotating the region enclosed by the graphs \( y = x^2 \), \( y = 12 - x \), and \( x = 0 \) about the axis \( y = 15 \).

Solution. Solving \( x^2 = 12 - x \) gives \( x = 3 \) or \(-4\). We’ll take the \( x \geq 0 \) portion, since the \( x \leq 0 \) portion intersects the axis and is weird. (N.B., this problem is not very clearly written.) The distances are \( 15 - x^2 \) and \( 15 - 12 + x = 3 + x \), respectively. Thus the volume is given by \( V = \pi \int_{0}^{3} (15 - x^2)^2 - (3 + x)^2 \, dx = \pi \left[ x^5/5 - 31x^3/3 - 3x^2 + 216x \right]_{0}^{3} = 1953\pi/5 \). □
2.4. **Exercise 6.3.54.** Verify the formula

\[ \int_{x_1}^{x_2} (x - x_1)(x - x_2) \, dx = \frac{1}{6}(x_1 - x_2)^3. \]

Then prove that the solid obtained by rotating the shaded region above \( y = mx + c \) and below \( y^2 = ax + b \) about the \( x \)-axis has volume \( V = \frac{\pi}{6} BH^2 \), with \( B \) the base of the region and \( H \) the height.

**Solution.** The verification is a trivial process, and is left as an exercise to the student. Let \( x_1 \) and \( x_2 \) be the roots of \( f(x) = ax + b - (mx + c)^2 \), where \( x_1 < x_2 \). Notice that \( V = \pi \int_{x_1}^{x_2} f(x) \, dx \). Since \( x_1 \) and \( x_2 \) are roots, and the coefficient of \( x^2 \) is \(-m^2\), we get that \( f(x) = -m^2(x - x_1)(x - x_2) \). Thus the integral becomes \( V = \pi \int_{x_1}^{x_2} (-m^2)(x - x_1)(x - x_2) \, dx \). By the formula given, we get \( V = -m^2 \pi \cdot \frac{4}{3}(x_1 - x_2)^3 \).

Notice that \( x_2 - x_1 = B \) and \( m = \frac{H}{B} \), we finally get \( V = \frac{\pi}{6} BH^2 \), as desired. \( \square \)