1. Differentiation

1.1. Basics. Given a function \( f(x) \). The slope of the tangent line at \( x = c \) is \( f'(c) \).

1.1.1. Higher Derivatives. Recursively define higher derivatives: \( f^{(n)} = (f^{(n-1)})' \).

1.1.2. Chain Rule. If \( f \) and \( g \) are differentiable, then \((f \circ g)(x) = f(g(x))\) is differentiable and \((f(g(x)))' = f'(g(x))g'(x)\).

1.2. Exercise 3.6.51. Show that a nonzero polynomial function \( y = f(x) \) cannot satisfy the equation \( y'' = -y \). Use this to prove that neither \( \sin x \) nor \( \cos x \) is a polynomial.

Proof. Let \( y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \), with \( a_n \neq 0 \). If the degree \( n < 2 \), then \( y'' \equiv 0 \) so \( y'' \neq -y \). Otherwise, \( y' = na_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \ldots + 2a_2 x + a_1 \), and \( y'' = (n-1)na_n x^{n-2} + \ldots + 2a_2 \). Since \( y'' \) lacks a monomial \( x^n \), we cannot have \( y'' = -y \). \( \square \)

1.3. Exercise 3.7.92. A Discontinuous Derivative. Use the limit definition to show that \( g'(0) \) exists but \( g'(0) \neq \lim_{x \to 0} g'(x) \), where \( g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \)

Proof. Recall \( g'(x) = \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} \). So by definition, \( g'(0) = \lim_{h \to 0} \frac{g(h)-g(0)}{h} = \lim_{h \to 0} h \sin \frac{1}{h} \). Using Squeeze Theorem and \(-|h| \leq h \sin \frac{1}{h} \leq |h|\), we get that \( g'(0) = 0 \). Away from \( x = 0 \), we can use the formula and get \( g'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot (-1) \cdot \frac{1}{x^2} = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \). Now \( \lim_{x \to 0} g'(x) \) does not exist since \( \lim_{x \to 0} 2x \sin \frac{1}{x} = 0 \) by Squeeze Theorem but \( \lim_{x \to 0} \cos \frac{1}{x} \) does not exist. \( \square \)

1.4. Exercise 3.8.35. Implicit Differentiation. If the derivative \( dx/dy \) exists at a point and \( dx/dy = 0 \), then the tangent line is vertical. Calculate \( dx/dy \) for the equation \( y^4 + 1 = y^2 + x^2 \) and find the points on the graph where the tangent line is vertical.

Solution. By implicit differentiation, we get \( 4y^3 = 2y + 2x \frac{dx}{dy} \). Setting \( dx/dy = 0 \), we get \( 4y^3 = 2y \), so \( y = 0, \pm \frac{1}{\sqrt{2}} \). If \( y = 0 \) then \( x = \pm 1 \), if \( y = \pm \frac{1}{\sqrt{2}} \), then \( x^2 = \frac{1}{2} \), so we get 6 points. \( \square \)
1.5. Exercise 3.9.44. Related Rates. A wheel of radius $r$ is centred at the origin. As it rotates, the rod of length $L$ attached at the point $P$ drives a piston back and forth in a straight line. Let $x$ be the distance from the origin to the point $Q$ at the end of the rod.

(a) Use the Pythagorean Theorem to show that

$$L^2 = (x - r \cos \theta)^2 + r^2 \sin^2 \theta.$$ 

(b) Differentiate part (a) with respect to $t$ to prove that

$$2(x - r \cos \theta) \left( \frac{dx}{dt} + r \sin \theta \frac{d\theta}{dt} \right) + 2r^2 \sin \theta \cos \theta \frac{d\theta}{dt} = 0.$$ 

(c) Calculate the speed of the piston when $\theta = \frac{\pi}{2}$, assuming that $r = 10$ cm, $L = 30$ cm, and the wheel rotates at 4 revolutions per minute.

Solution. Parts (a) and (b) are straightforward. 4 revolutions per minute means $\frac{d\theta}{dt} = 4 \cdot 2\pi$ per minute. From part (a), we get $30^2 = x^2 + 10^2$, so $x = 20\sqrt{2}$. Plugging in, we get $2(20\sqrt{2} - 0)(\frac{dx}{dt} + 10 \cdot 8\pi) + 0 = 0$. So $\frac{dx}{dt} = -80\pi$ cm per minute. □

1.6. Exercise 3.8.55. Lemniscate Curve. The lemniscate curve $(x^2 + y^2)^2 = 4(x^2 - y^2)$ was discovered by Jacob Bernoulli in 1694, who noted that it is “shaped like a figure 8, or knot, or the bow of a ribbon.” Find the coordinates of the four points at which the tangent line is horizontal.

Solution. By implicit differentiation and chain rule, we get $2(x^2 + y^2)(2x + 2yy') = 4(2x - 2yy')$. If $y' = 0$, we get $2(x^2 + y^2)(2x) = 4(2x)$, yielding $x^2 + y^2 = 2$. Substituting into the lemniscate curve, we get $x^2 - y^2 = 1$. So $x^2 = 3/2$ and $y^2 = 1/2$. □