

Equilibria and Efficiency Loss in Games on Networks

Joshua R. Davis*, Zachary Goldman[†], Jacob Hilty*, Elizabeth N. Koch*,
David Liben-Nowell*, Alexa Sharp[‡], Tom Wexler^{†‡}, and Emma Zhou*

*Carleton College; Northfield, MN [†]Denison University; Granville, OH [‡]Oberlin College; Oberlin, OH

joshuadavis@q.com, goldma_z@denison.edu, jacob.a.hilty@gmail.com, elizabeth.koch@alumni.carleton.edu,
dlibenno@carleton.edu, alexa.sharp@oberlin.edu, tom.wexler@oberlin.edu, zhoue@carleton.edu

Abstract—Social networks are the substrate upon which we make and evaluate many of our daily decisions: our costs and benefits depend on whether—or how many of, or which of—our friends are willing to go to that restaurant, choose that cellular provider, already own that gaming platform. Much of the research on the “diffusion of innovation,” for example, takes a game-theoretic perspective on strategic decisions made by people embedded in a social context. Indeed, multiplayer games played on social networks, where the network’s nodes correspond to the game’s players, have proven to be fruitful models of many natural scenarios involving strategic interaction.

In this paper, we embark on a mathematical and general exploration of the relationship between 2-person strategic interactions (a “base game”) and a “networked” version of that same game. We formulate a generic mechanism for superimposing a symmetric 2-player base game M on a social network G : each node of G chooses a single strategy from M and simultaneously plays that strategy against each of its neighbors in G , receiving as its payoff the sum of the payoffs from playing M against each neighbor. We denote the *networked game* that results by $M \oplus G$. We are broadly interested in the relationship between properties of M and of $M \oplus G$: how does the character of strategic interaction change when it is embedded in a social network? We focus on two particular properties: the (pure) price of anarchy and the existence of pure Nash equilibria. We show tight results on the relationship between the price of anarchy in M and $M \oplus G$ in coordination games. We also show that, with some exceptions when G is bipartite, the existence or absence of pure Nash equilibria (and even the guaranteed convergence of best-response dynamics) in M and $M \oplus G$ are not entailed in either direction. Taken together, these results suggest that the process of superimposing M on a graph is a nontrivial operation that can have rich, but bounded, effects on the strategic environment.

All authors were supported in part by NSF grant CCF-0728779. This work was also supported by grants from Carleton College, Oberlin College, and Denison University. Thanks to Michael Brooks, Katie Kuksenok, Eric Naeseth, and Charles Noneman for helpful discussions and comments.

I. INTRODUCTION

In recent years there has been significant and growing interest in games played on networks, where the game’s players are represented by the nodes of the network. Within this space, there has been particular attention paid to social networks, in which edges connect pairs of people whose actions can directly impact each other. This growing line of research can in part be attributed to a sense that such games are indeed the “right” model for many natural scenarios: many human endeavors can be viewed as games in which a person’s utility is determined by the behavior of those who are in some sense close by—friends, associates, or trading partners, for example. The actions of other players in the network are felt only indirectly; their decisions may or may not cause cascades of local changes that eventually propagate to distant players.

A rich vein of research, and an illustrative example of this game-theoretic style of work on social networks, has been carried out under the “diffusion of innovation” rubric in the literature (see [5, 15–18, 28, 31–33, 35], e.g.). Imagine, for example, a population of mobile phone users, each of whom must choose whether to subscribe to an unlimited text-messaging plan with her cellular provider, or to use a pay-per-message plan. If many of a person u ’s friends adopt an unlimited-texting plan, then u may receive many (perhaps unsolicited) text messages from her friends, incurring a large number of high pay-per-message charges; conversely, if very few of u ’s friends have adopted the unlimited-message option, then u will likely receive few text messages, and thus the pay-per-use price will likely be a better deal. This scenario requires people to think not only about the relative costs of “renting” and “buying” but also about the structure of their social network, and the

choices that their friends make. In this example, one might expect to observe clusters of adopters in the social network that are relatively isolated from corresponding clusters of nonadopters. In reality, these clusters are highly correlated with age, and in part result from the significant correlation in age between people connected by a social tie (see [27]). This age homophily illustrates the perhaps-obvious point that in many network-type games the structure of the network matters critically. But, perhaps also obviously, the structure of the game matters too. Sometimes a strategy (adopting a new piece of communication technology, say) becomes more attractive as additional people, particularly one’s friends, choose it; sometimes it becomes less attractive (ordering a dish at a restaurant where food will be shared) [21, 26, e.g.]. And more complex strategic landscapes are possible too.

In this paper, we will be broadly concerned with the way in which local decision making—whether to buy the unlimited text-messaging plan, whether to order the pad thai or the massaman curry, whether to go to the opera or the baseball game—is affected when it is embedded into a social-network context. We formulate a general framework for superimposing a symmetric 2-player game, which we call a *base game*, on a social network, creating a networked version of the base game. The nodes of the network correspond to the players, each of whom must choose a single strategy from the set of available strategies in the base game. Each player’s payoff is the sum of its payoffs from playing the base game simultaneously with its neighbors. (In essence, each player simultaneously plays the base game with each of its neighbors in the graph, restricted so that players must act consistently across their multiple games.) Our particular line of inquiry is directed towards exploring the relationship among a base game M , a graph G , and the resulting networked game $M \oplus G$.

Our results: We study the networked games resulting from superimposing arbitrary 2-player symmetric games on arbitrary undirected graphs. We are primarily interested in understanding how properties of a base game carry over—or do not carry over—to the networked game. There are many ways to ask about the relationship among M , G , and $M \oplus G$. We focus on two specific instantiations of this question, as initial steps in this line of study: the (pure) price of anarchy and the existence of (pure) Nash equilibria. See Section III for definitions. (Throughout, we consider only pure strategies except where mixed strategies are explicitly mentioned; see Sections V and VII for discussion of mixed strategies.)

- **The price of anarchy.** The (*pure*) price of anarchy of a game is the ratio of the social welfare—the sum

of the utilities for all players—in the best outcome OPT to the worst Nash equilibrium WNE. We are interested in how the price of anarchy of M relates to the price of anarchy of $M \oplus G$. Here we focus on *coordination games* as base games: players receive identical positive payoffs if they choose the same strategy from the set of options, and payoffs of zero if they choose different strategies. (The payoffs the players receive for matching can depend on which strategy they choose.) Coordination games and variants are frequently used to model diffusion of innovation: for example, strategies might represent choices of communication technologies where there is utility in making the same choice as one’s friends. Let M_{coord} be any coordination game of this type. In Section V, we give tight bounds on the maximum (taken over all graphs) of the price of anarchy for $M_{\text{coord}} \oplus G$.

- **The existence of pure Nash equilibria.** One of the most basic game-theoretic questions that one can ask is whether a pure Nash equilibrium exists in a given game. The analogous question here is the connection between the existence of pure Nash equilibria in the base game and the existence of pure Nash equilibria in the networked game. In Section VI, we show largely negative results about this connection. If M has a pure Nash equilibrium and G is bipartite, then $M \oplus G$ has a pure Nash equilibrium as well; in all other cases, though, we show that the existence (or absence) of pure Nash equilibria in M does not imply anything about the existence (or absence) of pure Nash equilibria in $M \oplus G$. We give examples of base games M and (bipartite and nonbipartite) graphs G_1 and G_2 such that the properties of M and $M \oplus G_1$ match with respect to the existence of pure Nash equilibria, but M and $M \oplus G_2$ mismatch. Furthermore, we give examples of base games M in which best-response dynamics is guaranteed to converge in M but is not guaranteed to converge in $M \oplus G$, and vice versa.

Taken together, these results suggest that “network-ifying” a base game is a nontrivial operation that can have rich effects on the strategic environment. Pure Nash equilibria can be created or destroyed; the guaranteed convergence of best-response dynamics can be introduced or eliminated. Still, moving a base game into a networked setting cannot have an unbounded effect on its properties, as the results on the price of anarchy for coordination games show. We believe that further quantification of the effect of moving a base game to a networked context is a fertile area for study.

II. RELATED WORK

As described in Section I, there has been appreciable work on networked versions of particular games, usually on particular classes of networks, in modeling the diffusion of innovation [5, 15–18, 28, 31–33, 35]. These models typically capture scenarios in which there is incentive towards assortative behavior; other work has explored models where players have incentive towards disassortative behavior, including *cut games*, and *party affiliation games*, which subsume both cut games and coordination games [1, 4, 10].

A networked game $M \oplus G$ is a special form of a *graphical game* [3, 6–9, 13, 20, 22, 24], an n -player game in which the payoff to a player u is affected only by the strategies of the neighbors of u in an underlying n -node graph. (Other formalisms for games on networks have also been considered; see [12, 14, 19] for some examples.) Graphical games allow u 's payoff to depend arbitrarily on the strategies chosen by u 's neighbors; in our networked games u 's payoff is simply the sum of a payoff on each edge incident to u . (Graphical games are interesting when the underlying graph is sparse, as social networks are; otherwise the graphical structure does not impose much limitation.) Our networked games, as well as these “sparse” graphical games, form a natural class of *compactly representable games*—games that can be specified in space polynomial in the number of players and strategies. Fully general games are typically of less interest, both practically and theoretically: practically, general games require exponential space to describe and thus are too large to be tractable; and, theoretically, this huge game description trivializes various computational problems—searching for a pure Nash equilibrium in a general game can be solved by brute force in time linear in the input size.

One paper on graphical games is particularly close in spirit to our work here: Ben Zwi and Ronen [3] study the relationship between the (“global”) price of anarchy of a graphical game and what they call the “local price of anarchy.” The latter measures how well any subset S of players in the network responds to choices made by the nodes outside of S (where “how well” is measured in terms of WNE/OPT within the subgame induced by fixing the strategies of all non- S players).

At a more abstract level, one can see a parallel between our work here and recent work in epidemiology. Work in that field can loosely be categorized as falling under the “fully mixed model”—in which one models any two members of a large population as equally likely to interact—or under the “network model,” in which an underlying social/contact network reflects the latent

structure in people’s interactions. Network-based models more realistically reflect the ways in which diseases spread, but in most cases the fully mixed model is far more amenable to rigorous mathematical analysis, particularly as the population grows large. In the game-theoretic context, there is a significant body of work on “fully mixed”-type models in which randomly selected pairs of agents from a large population play a particular two-player game, possibly repeatedly—for example, see the classic text of Fudenberg and Levine on learning in games [11]. (Their motivations include giving an account of how players might settle on a Nash equilibrium, and how players might select a particular equilibrium.) In our setting, we consider a fixed network of interactions; our interests are in the ways that the networked game relates to the base game. This network-based perspective on evolutionary game theory was introduced in the work of Kearns and Suri [23], who were largely interested in extending classic results of evolutionary game theory to the network-based setting.

III. BACKGROUND AND NOTATION

Background on game theory: An n -player game M consists of a set of *players* $\{1, \dots, n\}$, a *strategy set* S_i for each player i , and a *payoff function* $p_i : (S_1 \times \dots \times S_n) \rightarrow \mathbb{Z}$ for each player. Each player i chooses a strategy $s_i \in S_i$; the vector $\mathbf{s} = \langle s_1, \dots, s_n \rangle$ will be called a *strategy profile*. The *payoff* to player i under \mathbf{s} is $p_i(\mathbf{s})$, and the *social welfare* of \mathbf{s} is $\sum_i p_i(\mathbf{s})$. The *social optimum*, denoted OPT, is the strategy profile \mathbf{s} that achieves the maximum social welfare. A strategy profile \mathbf{s} is a (*pure*) *Nash equilibrium* if no player can unilaterally deviate from \mathbf{s} to improve her payoff, i.e., if $p_i(\mathbf{s}) \geq p_i(s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n)$ for every i and every $s'_i \in S_i$. We denote the worst Nash equilibrium—that is, the pure Nash equilibrium \mathbf{s} with the lowest social welfare—by WNE, and the best Nash equilibrium by BNE. We write OPT, WNE, and BNE to also denote the social welfare of the corresponding strategy profile OPT, WNE, and BNE. The *price of anarchy* is given by $\text{POA} := \text{OPT}/\text{WNE}$, and the *price of stability* is given by $\text{POS} := \text{OPT}/\text{BNE}$. Note again that all of these quantities refer only to pure Nash equilibria.

In this paper, we are interested in *symmetric* two-player games, where the two players have the same set of strategies $S_1 = S_2 = S$, and $p_1(s, s') = p_2(s', s)$ for any two strategies $s, s' \in S$. Whenever we refer a “base game,” we implicitly mean it to be a symmetric two-player game.

Graphs: For the purposes of this paper, a *graph* $G = \langle V, E \rangle$ is an undirected graph that has no isolated nodes. We write \mathcal{K}_n to denote the n -node complete graph,

and $\mathcal{K}_{n,n}$ to denote the complete bipartite graph with n nodes in each “part” of the graph.

IV. PLAYING GAMES ON GRAPHS

Let M be an arbitrary symmetric two-player game where each player’s set of available strategies is denoted by S , and when player 1 plays s_1 and player 2 plays s_2 then the payoff to player i is given by $p_i(s_1, s_2)$. Let $G = \langle V, E \rangle$ be an arbitrary graph. We define the *networked game* $M \oplus G$ (“game M played on graph G ”) as follows:

- The set of players in $M \oplus G$ is V , the set of nodes in the graph G .
- The set of strategies available to player $v \in V$ is S , the set of strategies for a player in M .
- For a strategy profile \mathbf{s} in which player v plays the strategy $s_v \in S$, the payoff to any player $w \in V$ is given by $p_w(\mathbf{s}) := \sum_{x \in \Gamma(w)} p_1(s_w, s_x)$, where $\Gamma(w)$ denotes the set of neighbors of node w in G .

For example, if G is a two-node, one-edge graph (\mathcal{K}_2), then $M \oplus G$ is isomorphic to the base game M .

We can see a less trivial example through rock–paper–scissors. We define the base game RPS with strategies {rock, paper, scissors} such that $p_1(\text{rock}, \text{scissors}) = p_1(\text{scissors}, \text{paper}) = p_1(\text{paper}, \text{rock}) = 1$ and all other payoffs are zero. This game has no pure Nash equilibria. But in $\text{RPS} \oplus \mathcal{K}_3$, when this game is played on the triangle, we have a three-player game that does have a pure Nash equilibrium—namely, one where each of the three strategies is played by exactly one player.

V. COORDINATION GAMES

We begin by analyzing the networked version of coordination games. By way of reminder, our focus in this section (and indeed throughout the paper) is on the mathematical properties of these network games—particularly as those properties relate to their analogues in the base coordination game. In this section, we focus on price-of-anarchy and price-of-stability results.

A two-player *coordination game* (or *matching game*, or *consensus game*) is one in which both players choose from the same set of strategies, and receive positive payoffs only if they make the same selection. As usual, we are interested in symmetric coordination games, which we define as follows:

- Players share a strategy set $S = \{a_1, \dots, a_k\}$.
- The two players both receive identical utilities of v_i if both choose a_i for any index $i \in \{1, \dots, k\}$, and both receive identical utilities of 0 if they choose strategies a_i and $a_{j \neq i}$.

We refer to this game as the *k-strategy coordination game with payoffs* $\{v_1, \dots, v_k\}$. Without loss of generality, we always canonically order the strategies so that

$v_1 \leq v_2 \leq \dots \leq v_{k-1} \leq v_k$. Throughout, we assume that the v_i ’s are integer-valued.

Lemma 1. *Let M be an arbitrary k-strategy coordination game with payoffs $\{v_1, \dots, v_k\}$. Then:*

- 1) $\text{POS}(M) = \text{POS}(M \oplus G) = 1$ for any graph G .
- 2) $\exists G : \text{POA}(M \oplus G) = v_k \cdot \sum_{i=1}^k \frac{1}{v_i}$.
- 3) $\text{POA}(M \oplus G) \leq v_k \cdot \sum_{i=1}^k \frac{1}{v_i}$ for any graph G .

Proof sketch (details omitted for reasons of space):

Claim (1) is easy: the social optimum is when all nodes play strategy a_k , which is also a Nash equilibrium.

Claim (2) can be proven for a complete bipartite graph $\mathcal{K}_{n,n}$, for an appropriately chosen size n . Specifically, define $n_i := \prod_{j \neq i} v_j$, and let $n := \sum_i n_i$. We can show that the worst Nash equilibrium is formed if precisely n_i of the n nodes in each part of $\mathcal{K}_{n,n}$ play strategy a_i . This Nash equilibrium’s social welfare is a factor of $v_k \cdot \sum_i \frac{1}{v_i}$ worse than OPT, and Claim (2) follows.

To prove Claim (3), fix an arbitrary Nash equilibrium strategy profile \mathbf{p} in $M \oplus G$. Write $\delta(u)$ to denote u ’s degree in G ; write $\delta^i(u)$ to denote the number of u ’s neighbors in G who play a_i under \mathbf{p} ; write $m = \sum_u \delta(u)/2$ to denote the total number of edges in G ; and write p_u to denote u ’s payoff under \mathbf{p} .

Note that u ’s payoff from playing strategy a_i is precisely $v_i \delta^i(u)$. For any u and any $i \in \{1, \dots, k\}$, we have $p_u \geq v_i \delta^i(u)$ because \mathbf{p} is a Nash equilibrium (otherwise u would prefer to play a_i). Dividing both sides by v_i and summing the resulting constraints over all strategies i , we have $\sum_i \frac{p_u}{v_i} \geq \sum_i \delta^i(u)$. But $\sum_i \delta^i(u)$ is just $\delta(u)$, so we have $p_u \cdot \sum_i \frac{1}{v_i} \geq \delta(u)$ and thus $p_u \geq \frac{\delta(u)}{\sum_i \frac{1}{v_i}}$. Therefore the total utility of \mathbf{p} is given by

$$\sum_u p_u \geq \sum_u \frac{\delta(u)}{\sum_i \frac{1}{v_i}} = 2m \cdot \frac{1}{\sum_i \frac{1}{v_i}}.$$

The social optimum is achieved when all players choose strategy a_k , achieving a payoff of $\text{OPT} = 2m \cdot v_k$. Thus the ratio of the social welfares of OPT and \mathbf{p} is at most $(2m \cdot v_k) / (2m / \sum_i \frac{1}{v_i}) = v_k \sum_i \frac{1}{v_i}$. Because \mathbf{p} was an arbitrary Nash equilibrium, we have that $\text{POA}(M \oplus G) \leq v_k \cdot \sum_i \frac{1}{v_i}$ and the claim follows. ■

Theorem 2. *Let M be any k-strategy coordination game, for any $k \geq 2$. Then*

$$\text{POA}(M) + k - 1 \leq \max_G \text{POA}(M \oplus G) \leq k \cdot \text{POA}(M).$$

Furthermore, for any k, there are k-strategy coordination games in which both bounds are tight.

Proof: Let M be an arbitrary k-strategy coordination game with payoffs $\{v_1, \dots, v_k\}$. Lemmas 1.2 and 1.3 imply that $\max_G \text{POA}(M \oplus G) = v_k \cdot \sum_i \frac{1}{v_i}$.

Let G^* denote the graph achieving this maximum price of anarchy. Also notice that $\text{POA}(M) = v_k/v_1$: in the base game, the only Nash equilibria are when the two players are indeed playing the same strategy, and the worst such equilibrium is when both players play a_1 .

For the lower bound, $\text{POA}(M \oplus G^*) = v_k \cdot \sum_i \frac{1}{v_i} \geq v_k \cdot (\frac{k-1}{v_k} + \frac{1}{v_1}) = k-1 + \frac{v_k}{v_1} = k-1 + \text{POA}(M)$, because $1/v_k \leq 1/v_i$ for all i . The bound is tight in the k -strategy coordination game with payoffs $\{1, v, v, \dots, v\}$, where $\text{POA}(M) = v/1 = v$ and $\text{POA}(M \oplus G^*) = k-1 + v$.

For the upper bound, $\text{POA}(M \oplus G^*) = v_k \cdot \sum_i \frac{1}{v_i} \leq v_k \cdot \frac{k}{v_1} = k \cdot \text{POA}(M)$, because $1/v_i \leq 1/v_1$ for all i . This bound is tight in the k -strategy coordination game M with payoffs $\{1, 1, \dots, 1\}$, where we have $\text{POA}(M) = 1$ while $\text{POA}(M \oplus G^*) = k$. ■

As we stated in Section I, in this paper we restrict our attention to pure Nash equilibria. In many of the scenarios that motivate our work, the choices participants make (e.g., which operating system to buy, or which cell phone provider to use) are costly to alter, and it is hard to imagine these decisions literally being made by coin tosses by every member of a large population.

That said, it is worth noting that the constructed pure Nash equilibrium on $M \oplus \mathcal{K}_{n,n}$ of Lemma 1.2 is closely related to a mixed Nash equilibrium in the base game M . In particular, we can associate Player 1 in M with all players on the left side of $\mathcal{K}_{n,n}$ in $M \oplus \mathcal{K}_{n,n}$, and Player 2 with all players on the right side. Given any pure Nash equilibrium in $M \oplus \mathcal{K}_{n,n}$, the corresponding mixed strategy for a player in M is a weighted average of her associated players' strategies. Similarly, any mixed Nash equilibrium on M can be converted into a pure Nash equilibrium in a networked game on a sufficiently large complete bipartite graph (assuming probabilities in the mixed Nash equilibrium are rational). See Theorem 6 for the general, formal version of this statement.

This correspondence implies the following surprising fact: Lemma 1.2 and Lemma 1.3 also relate the pure and mixed prices of anarchy in (two-player) coordination games. Of course, Lemma 1.3 is a more general result, in that it bounds the effect of playing matching games on *arbitrary* networks, not just complete bipartite graphs, and pure Nash equilibria in $M \oplus G$ for general networks G do not correspond (at least not obviously) to mixed Nash equilibria in M . Nevertheless, as part of our future work we intend to further explore the relationship between mixed Nash equilibria in base games and pure Nash equilibria in their networked counterparts.

VI. EXISTENCE OF PURE NASH EQUILIBRIA

In this section, we explore the relationship between the existence of pure Nash equilibria in an arbitrary base

game M and the existence of pure Nash equilibria in the networked game $M \oplus G$ played on a graph G .

We also further explore the case when there are pure Nash equilibria in M , by subdividing this case based on whether *best-response dynamics (BRD)* always converges. BRD is an algorithm that produces a sequence of strategy profiles by repeatedly allowing a player i to update her strategy to a *best response*, i.e., a strategy in S_i maximizing i 's payoff, holding all other strategies constant. Specifically, BRD is the following algorithm:

- 1) Start from an arbitrary strategy profile \mathbf{s} .
- 2) While a player is not playing a best response in \mathbf{s} :
 - a) Choose such a player i arbitrarily.
 - b) Update \mathbf{s} by changing s_i to be an arbitrary best response for player i .

If BRD terminates, then it terminates at a pure Nash equilibrium; however, BRD may not terminate even if a pure Nash equilibrium exists. We will say that BRD *always converges* if this process terminates regardless of the arbitrary choices (the initial profile \mathbf{s} , which player i updates in each iteration, which best response s_i is chosen if there are more than one), subject to the liveness condition that every player “gets a turn” infinitely often.

Our results are the following. It turns out that there is a natural division in our results based on whether the graph G is bipartite, so we distinguish these cases below.

- The absence of pure Nash equilibria in M does not imply anything about the existence of pure Nash equilibria in $M \oplus G$. The existence of pure Nash equilibria in M also does not imply anything about the existence of pure Nash equilibria in $M \oplus G$ for nonbipartite graphs G . However, for all bipartite graphs G , the existence of a pure Nash equilibrium in M does imply that a pure Nash equilibrium also exists in $M \oplus G$.
- When BRD always converges in M , we cannot conclude anything about the convergence of BRD in $M \oplus G$. We do prove the converse for bipartite graphs: if G is bipartite and BRD always converges in $M \oplus G$, then BRD always converges in M . However, for nonbipartite graphs, neither implication holds; there are examples in which BRD always converges in $M \oplus G$ but BRD does not always converge in M , and vice versa.

Our results are summarized in Figure 1. Due to space constraints, most proofs are sketched or omitted entirely.

In several of our constructions, we will make use of the following k -strategy “rock–paper–scissors” game, which we denote by RPS_k . There are $k \geq 2$ strategies $\{a_0, \dots, a_{k-1}\}$. Strategy a_i “beats” $a_{(i+1) \bmod k}$ to earn

	no pure Nash equil. exists in M	a pure Nash equilibrium exists in M	
		BRD does not converge in M	BRD converges in M
bipartite graph G	$\forall G \exists M$: no pure NE exists (Thm. 3)	$\forall G \forall M$: a pure NE exists (Theorem 4)	
	$\exists G \forall M$:	$\forall G \forall M$: BRD loops (Thm. 8)	$\exists M \forall G$: BRD converges (Thm. 7) $\exists G \forall M$: BRD converges (Thm. 7) $\exists G \exists M$: BRD loops (Thm. 9)
nonbipartite graph G	$\forall M \exists G$: no pure NE exists (Thm. 7)	$\forall G \exists M$: no pure NE exists (Theorem 5) $\forall M \exists G$: a pure NE exists (Theorem 6)	
	$\forall M \exists G$: a pure NE exists (Thm. 6)	$\forall G \exists M$: BRD loops (Thm. 5) $\exists G \forall M$: BRD loops (Thm. 7) $\exists G \exists M$: BRD converges (Thm. 9)	$\forall G \exists M$: BRD loops (Thm. 5) $\exists M \forall G$: BRD converges (Thm. 7)

Fig. 1. Summary of our results relating the existence of pure Nash equilibria and the convergence of best-response dynamics (BRD) in base games M and networked games $M \oplus G$. When we write “BRD converges” we mean that BRD *always* converges, from any starting strategy profile and following any sequence of best-response updates. Similarly, “BRD loops” means that BRD *sometimes* fails to converge.

a unit payoff; all other payoffs are zero. For example:

$$\text{RPS}_5 = \begin{array}{c} \begin{array}{ccccc} & a_0 & a_1 & a_2 & a_3 & a_4 \\ a_0 & 0, 0 & 1, 0 & 0, 0 & 0, 0 & 0, 1 \\ a_1 & 0, 1 & 0, 0 & 1, 0 & 0, 0 & 0, 0 \\ a_2 & 0, 0 & 0, 1 & 0, 0 & 1, 0 & 0, 0 \\ a_3 & 0, 0 & 0, 0 & 0, 1 & 0, 0 & 1, 0 \\ a_4 & 1, 0 & 0, 0 & 0, 0 & 0, 1 & 0, 0 \end{array} \end{array}.$$

For any scalar $\alpha > 0$, write $\alpha \cdot \text{RPS}_k$ to denote the rescaled version of RPS_k where the positive payoffs are α instead of 1.

Note that RPS_k does not have a pure Nash equilibrium for any $k \geq 3$: the only best response to strategy a_i is $a_{(i-1) \bmod k}$, but the only best response to $a_{(i-1) \bmod k}$ is $a_{(i-2) \bmod k}$. Thus no two strategies are mutual best responses—and thus no pure Nash equilibrium exists—because $i \not\equiv i-2 \pmod{k}$ for any $k \geq 3$. Incidentally, for $k = 2$, it is the case that $i \equiv i-2 \pmod{k}$, and thus that a_0 and a_1 are mutual best responses. Indeed, RPS_2 is a simple “mismatching” game, with pure Nash equilibria $\langle a_0, a_1 \rangle$ and $\langle a_1, a_0 \rangle$. In fact, the networked form of RPS_2 yields the “cut games” described briefly at the beginning of Section II.

Theorem 3. *For all graphs G , there exists a base game M_G with no pure Nash equilibria such that $M_G \oplus G$ also does not have a pure Nash equilibrium.*

Proof sketch: Let $M_G = \text{RPS}_{n+1}$, where n is the number of nodes in G . There are more strategies than players, and thus in any pure strategy profile \mathbf{p} there is some node u who is not beating any of its neighbors; thus u is not playing a best response in \mathbf{p} . ■

The proof of Theorem 3 uses a base game M with no pure Nash equilibria to show that $M \oplus G$ need not have a pure Nash equilibrium. If we restrict the base game M to contain a pure Nash equilibrium, will the same result hold? The answer turns out to depend on the structure of G , specifically whether G is bipartite.

Theorem 4. *For all bipartite graphs G and for all base games M with a pure Nash equilibrium, the game $M \oplus G$ has a pure Nash equilibrium too.*

Proof sketch: Take a pure Nash equilibrium $\langle a, b \rangle$ for M , and have all nodes in the left “part” of G play a and all nodes in the right part play b . ■

Theorem 5. *For all nonbipartite graphs G , there exist base games M, M' , both with pure Nash equilibria, but neither $M \oplus G$ nor $M' \oplus G$ has a pure Nash equilibrium. Furthermore, BRD always converges in M and does not always converge in M' .*

Proof sketch: Let n be the number of nodes in G . Define the $(2n+2)$ -strategy game M_n as shown in Fig. 2. We can show that BRD always converges in M_n (at most one best-response update results in a Nash equilibrium). However, there is no pure Nash equilibrium in $M_n \oplus G$. Our proof rests on the existence of a connected set of ≥ 2 nodes that are all a -players or b -players (which must exist because G is nonbipartite) and an argument that they cannot all be playing best responses simultaneously, similar to Theorem 3.

M' is M_n augmented with two strategies c_0 and c_1 , where playing c_i against any non- c strategy has a highly negative payoff, and the c -versus- c submatrix is a version of the classic Matching Pennies game (a 2-player,

$$M_n :=$$

	a_0	a_1	\dots	a_n	b_0	b_1	\dots	b_n
a_0	$(1 - \varepsilon) \cdot \text{RPS}_{n+1}$				1, 1	1, 1	\dots	1, 1
a_1					1, 1	1, 1	\dots	1, 1
\vdots					\vdots	\vdots	\ddots	\vdots
a_n					1, 1	1, 1	\dots	1, 1
b_0	1, 1	1, 1	\dots	1, 1	$(1 - \varepsilon) \cdot \text{RPS}_{n+1}$			
b_1	1, 1	1, 1	\dots	1, 1				
\vdots	\vdots	\vdots	\ddots	\vdots				
b_n	1, 1	1, 1	\dots	1, 1				

Fig. 2. The construction for Theorem 5 of a base game that has a pure Nash equilibria but the networked game does not.

2-strategy game with no pure Nash equilibria). This construction blocks BRD's convergence in M' without creating additional equilibria in $M' \oplus G$. ■

In Theorems 4 and 5 we see how base games with pure Nash equilibria can lead to networked games with and without pure Nash equilibria, depending on the underlying graph structure. Theorem 3 shows a specific base game with no pure Nash equilibria that leads to a networked game with no pure Nash equilibria. But in Theorem 3, the base game M was chosen with respect to a particular graph G . Here we show that this dependence was crucial: for every base game M , there exists a graph G so that $M \oplus G$ has a pure Nash equilibrium. Our proof again highlights some of the connections between pure Nash equilibria in networked games and mixed Nash equilibria in base games:

Theorem 6. *For every base game M , there exist a bipartite graph G and a nonbipartite graph G' such that pure Nash equilibria exist in both $M \oplus G$ and $M \oplus G'$.*

Proof sketch: Let S denote the strategy set of M . A classic result of Nash [29] says that M has a symmetric mixed Nash equilibrium—that is, a probability distribution σ over S that is a best response to itself. Furthermore the distribution σ contains only rational probabilities, so that $\sigma(s) = q_s/n$, where $\sum_s q_s = n$, for a common denominator $n \in \mathbb{Z}^{\geq 1}$ and for coefficients $q_s \in \mathbb{Z}^{\geq 0}$. (See [30].) Our graphs are the complete bipartite graph $G := \mathcal{K}_{n,n}$ and the complete tripartite graph $G' := \mathcal{K}_{n,n,n}$ with n nodes in each part of each graph. We can show that a pure Nash equilibrium is formed in the networked game if precisely q_s nodes in each part of the graph play strategy s , for each s . ■

We also note the existence of simple examples of

graphs and base games in which properties of the base game do align with the corresponding networked games:

Theorem 7. (a) *There exists a base game M such that, for every graph G , BRD always converges in both the base game M and the networked game $M \oplus G$.*

(b) *There exists a bipartite graph G so that the existence of pure Nash equilibria and the convergence of BRD are identical in M and $M \oplus G$, for any M .*

(c) *There exists a nonbipartite graph G' so that, for every M , (i) a pure Nash equilibrium exists in $M \oplus G'$ only if a pure Nash equilibrium exists in M ; and (ii) BRD always converges in $M \oplus G'$ only if BRD always converges in M .*

Finally, we conclude this section by mentioning three results about best-response dynamics. The first result is analogous to Theorem 4, where we showed the existence of pure Nash equilibria in a base game carried over to networked games on bipartite graphs:

Theorem 8. *For every bipartite graph G and every M in which BRD does not always converge, BRD also does not always converge in $M \oplus G$ (even if players' best-response updates are done in round-robin order).*

We end with two more examples of mismatches in properties between base games and networked games:

Theorem 9. (a) *There exists a bipartite graph G and a base game M where BRD always converges in M but does not always converge in $M \oplus G$ (even if players' best-response updates are done in round-robin order).*

(b) *There exists a nonbipartite graph G and a base game M , where M has a pure Nash equilibrium but BRD does not always converge in M , such that BRD does always converge in $M \oplus G$.*

VII. FUTURE DIRECTIONS

Our work points to a substantial number of (we believe) interesting open questions. We are currently pursuing a number of these directions, which we briefly highlight here.

First, we would like to extend our price-of-anarchy analysis of coordination games to other or more general base games. What properties must a base game possess for similar bounds to hold?

Second, we hope to develop stronger results based on particular structural aspects of the underlying network. A number of our results in this paper distinguish between graphs that are bipartite and those that are not. Could stronger results be found if we restrict ourselves to other classes of graphs, such as regular graphs, trees, or grids? Or, more ambitiously, could we perhaps begin to

understand games played on networks that are explicitly intended to represent social structures (e.g., [2, 25, 34])?

Third, in Section V and in Theorem 6, we noted a correspondence between mixed Nash equilibria in base games and pure Nash equilibria in their networked counterpart (for a complete k -partite network). Can a more general correspondence be found for arbitrary networks? And could this correspondence further our understanding of mixed Nash equilibria in general?

Fourth, what kind of general algorithmic results are possible in the context of networked games? For example, can we give an efficient algorithm that, given a graph G as input, computes the price of anarchy for a simple coordination game when it is played on the graph G ? Under what circumstances can one find pure Nash equilibria efficiently in networked games? The connection to graphical games may be helpful here, though there are differences. (For example, even for a complete graph G , the game $M \oplus G$ is still compactly representable; graphical games where the underlying graph of direct influence is complete require an exponential-sized description.) Because our $M \oplus G$ networked games are more restrictive than graphical games, we might hope for efficient algorithms in a broader context than what is known for graphical games.

Finally, the framework we put forth in this work specifies how a large, complex game can be generated via the composition of a simple base game and a network. To what extent might we be able to perform the reverse operation: take a complex game and decompose it (even if only approximately) into a network and one or more simple base games?

REFERENCES

- [1] M.-F. Balcan, A. Blum, and Y. Mansour, "Improved equilibria via public service advertising," in *Proc. Symp. on Discrete Algorithms (SODA)*, 2009, pp. 728–737.
- [2] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, pp. 509–512, 1999.
- [3] O. Ben-Zwi and A. Ronen, "The local and global price of anarchy of graphical games," in *Proc. Symp. on Algorithmic Game Theory (SAGT)*, 2008, pp. 255–266.
- [4] G. Christodoulou, V. Mirrokni, and A. Sidiropoulos, "Convergence and approximation in potential games," in *Proc. Symp. on Theoretical Aspects of CS (STACS)*, 2006, pp. 349–360.
- [5] M. Chwe, "Communication and coordination in social networks," *Rev. of Econ. Studies*, vol. 67, no. 1, pp. 1–16, Jan. 2000.
- [6] C. Daskalakis and C. H. Papadimitriou, "Computing pure Nash equilibria in graphical games via Markov Random Fields," in *Proc. Conf. on Electronic Commerce (EC)*, 2006, pp. 91–99.
- [7] B. Dilkina, C. P. Gomes, and A. Sabharwal, "The impact of network topology on pure Nash equilibria in graphical games," in *AAAI Conference on Artificial Intelligence*, 2007, pp. 42–49.
- [8] E. Elkind, L. A. Goldberg, and P. W. Goldberg, "Nash equilibria in graphical games on trees revisited," in *Proc. Conf. on Electronic Commerce (EC)*, 2006, pp. 100–109.
- [9] —, "Computing good Nash equilibria in graphical games," in *Proc. Conf. on Electronic Commerce (EC)*, 2007, pp. 162–171.
- [10] A. Fabrikant, C. Papadimitriou, and K. Talwar, "The complexity of pure Nash equilibria," in *Proc. Symp. on Theory of Computing (STOC)*, 2004, pp. 604–612.
- [11] D. Fudenberg and D. K. Levine, *The Theory of Learning in Games*. MIT Press, 1998.
- [12] A. Galeotti, S. Goyal, M. Jackson, F. Vega-Redondo, and L. Yariv, "Network games," *Rev. of Econ. Studies*, 2009, to appear.
- [13] G. Gottlob, G. Greco, and F. Scarcello, "Pure Nash equilibria: hard and easy games," *Journal of Artificial Intelligence Research*, vol. 24, pp. 357–406, 2005.
- [14] S. Goyal, *Connections: An Introduction to the Economics of Networks*. Princeton University Press, 2007.
- [15] N. Immorlica, J. Kleinberg, M. Mahdian, and T. Wexler, "The role of compatibility in the diffusion of technologies through social networks," in *Proc. Conf. on Electronic Commerce (EC)*, 2007, pp. 75–83.
- [16] M. Jackson and L. Yariv, "Diffusion on social networks," *Économie Publique*, vol. 16, no. 1, pp. 3–16, 2005.
- [17] —, "Diffusion of behavior and equilibrium properties in network games," *American Economic Review*, vol. 97, no. 2, pp. 92–98, May 2007.
- [18] —, "Diffusion, strategic interaction, and social structure," in *Handbook of Social Economics*, J. Benhabib, A. Bisin, and M. O. Jackson, Eds. Elsevier, 2009, to appear.
- [19] M. O. Jackson, *Social and Economic Networks*. Princeton University Press, 2008.
- [20] S. Kakade, M. Kearns, J. Langford, and L. Ortiz, "Correlated equilibria in graphical games," in *Proc. Conf. on Electronic Commerce (EC)*, 2003, pp. 42–47.
- [21] M. L. Katz and C. Shapiro, "Network externalities, competition, and compatibility," *Am. Econ. Rev.*, vol. 75, no. 3, pp. 424–440, 1985.
- [22] M. Kearns, "Graphical games," in *Algorithmic Game Theory*, N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani, Eds. Cambridge University Press, 2007, pp. 159–178.
- [23] M. Kearns and S. Suri, "Networks preserving evolutionary equilibria and the power of randomization," in *Proc. Conf. on Electronic Commerce (EC)*, 2006, pp. 200–207.
- [24] M. J. Kearns, M. L. Littman, and S. P. Singh, "Graphical models for game theory," in *Proceedings of the Conference in Uncertainty in Artificial Intelligence (UAI)*, 2001, pp. 253–260.
- [25] J. Kleinberg, "The small-world phenomenon: An algorithmic perspective," in *Proc. Symp. on Theory of Computing (STOC)*, Jun. 2000, pp. 163–170.
- [26] S. J. Liebowitz and S. E. Margolis, "Network externality: An uncommon tragedy," *Journal of Economic Perspectives*, vol. 8, no. 2, pp. 133–150, Spring 1994.
- [27] M. McPherson, L. Smith-Lovin, and J. M. Cook, "Birds of a feather: Homophily in social networks," *Annual Review of Sociology*, vol. 27, pp. 415–444, Aug. 2001.
- [28] S. Morris, "Contagion," *Rev. of Econ. Studies*, vol. 67, pp. 57–78, 2000.
- [29] J. Nash, "Non cooperative games," *Annals of Mathematics*, vol. 54, pp. 286–295, 1951.
- [30] C. Papadimitriou, "The complexity of finding Nash equilibria," in *Algorithmic Game Theory*, N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani, Eds. Cambridge Univ. Press, 2007, pp. 29–51.
- [31] E. Rogers, *Diffusion of innovations*, 4th ed. Free Press, 1995.
- [32] D. Strang and S. Soule, "Diffusion in organizations and social movements: From hybrid corn to poison pills," *Annual Review of Sociology*, vol. 24, pp. 265–290, 1998.
- [33] T. Valente, *Network Models of the Diffusion of Innovations*. Hampton Press, 1995.
- [34] D. J. Watts and S. H. Strogatz, "Collective dynamics of 'small-world' networks," *Nature*, vol. 393, pp. 440–442, 1998.
- [35] H. P. Young, *Individual Strategy and Social Structure: An Evolutionary Theory of Institutions*. Princeton University Press, 1998.